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RICE CROP YIELD FORECASTING USING WEATHER PARAMETERS AT AN ORGANIZED FARM

Sachin Pathak*, V.K. Tyagiand S.B. Singh*****

(Received 11.07.2008 Revised 06.01.2009)

ABSTRACT

The present study is carried out at G. B. Pant University of Agriculture and Technology, Pantnagar, Uttarakhand, India, for developing forecast model based on weather variables for forecasting rice crop yield at an organized farm. Eight weather variables are taken as independent variables and crop yield is taken as dependent variable. With the help of these variables seventy two variables are generated taking correlation as weight. Forecast model have been developed using generated variables applying stepwise regression analysis. The study shows that crop yield can be forecasted at 17th week (about one and half month before harvest) with a high value of coefficient of multiple determination.

Key words: Weather parameters, Yield forecasting.

INTRODUCTION

Rice (*Oryza sativa* L.) is an important crop of India as well as of world. It is one of the main food grain crops grown in rainy season. The estimation of crop production is generally obtained by traditional methods after harvesting of crop. These estimates of crop yield are obtained through survey methods based on crop experiments and the area estimates are obtained by complete enumeration through administrative agencies or through other sources.

Estimates of crop production are needed by the farmers much before harvest for determining the prices of any agricultural commodity. These estimates are also very helpful in formation of import-export policies pertaining to agricultural produce, monitoring of advance action regarding procurement and distribution of food grains to entire planning process in the agricultural sectors. These estimates are also important to millers, traders and those who are engaged with the agribusiness activities.

Several studies have been carried out in India and abroad to examine the effect of weather variables on crop production. Many researchers in the past developed the forecast models based on weather parameters but most of them are limited to simple correlation and regression analysis though application of multiple regression technique is better compared to compressive analysis of crop weather relationship. Fisher [5] was the first person who worked on this topic. He handled this problem with the help of polynomial function. Hendrick and Schol [6] modified the Fisher's technique and they divided whole crop seasons into *n* weekly intervals and assumed that a second-degree polynomial in week number will be sufficiently

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flexible to express the relationship. The model suggested by Hendrick and Schol has been modified by Agrawal *et al.* [3] by expressing effect of changes in weather variables on yield in wth week as second degree polynomial in correlation coefficient between yield and weather variables of respective weeks. This model was further modified by Agrawal *et al.* [1] by expressing the effect of change in weather variables on yield in wth week as a linear function of respective correlation between yield and weather variables. But, these models have been developed based on limited time series data. To have a reliable forecast model a long time series data of 30-35 years is required. Thus there is a need for research work, which covers a long period of time and may explain the variability in crop production due to weather conditions at different growth stages. Keeping the above facts in view the present study is carried out to develop forecast model based on time series data of 30-35 years on weather variables and corresponding yield of rice crop on an organized farm at Pantnagar. The study is aimed to forecast crop yield at 17th week (about one and half month before harvest) with a high value of coefficient of multiple determination.

MATERIALS AND METHODS

1. **General description of study area:** The present study is carried out at Govind Ballabh Pant University of Agriculture and Technology Pantnagar, Udham Singh Nagar Uttarakhand, India. Pantnagar is situated at the Shivalik range of Himalayan foot hills. It is located at 29?N latitude and 79?27´E longitude and approximately 243.89 meters above mean sea level and lies in the Tarai belt of Uttarakhand. The soils of this region have good moisture holding capacity, about 7.3 PH and are highly productive. The annual rainfall of this region is about 1400 mm, which is subjected to large variation.

2. **Data:** Mainly two types of data are used in the present study.

2.1 Weather data: The present study is carried out on the weather data of 35 years (i.e., from 1966-67 to 2000-01). The data are collected from the Agro-meteorological observatory situated at Pantnagar University, which is equipped with modern technology and instruments. The weather variables which are under study are average weekly maximum temperature (X1), minimum temperature (X2), relative humidity at 07.00 hrs (X3), relative humidity at 14.00 hrs (X4), sun shine hrs (X5), weekly total rain fall (X6), number of rainy days per week (X7) and pan evaporation (X8). 22 weeks weather data are used for developing forecast model for the crops. The weather data are taken from May 21st (about a fortnight before sowing) to October, 22nd. The data of relative humidity have been transformed into arcsine root proportion as they are in percentage and square root transformation is used for data on number of rainy days per week.

2.2 Crop yield data: The crop yield data for same 33 years (from 1966-67 to 1998-99) are collected from the Pantnagar university farm which is a commercial farm. It is well irrigated, fully mechanized and well equipped with all modern agricultural technologies. At the farm all the high yielding varieties are grown, recommended doses of fertilizers and manure applied, proper plant protection measures and soil water management practices are under taken regularly. Therefore, we are assuming that under such a high level management the year to year variations in yield can largely be attributed to variation of weather conditions.

3. Crop growth stages: In Pantnagar rice crop is normally sown in the first week of June and harvested in the first week of November. The whole crop growth season can be divided into three important growth stages.

3.1 Seedling Stage: This period is from June 4th to July 1st. This stage includes the period from sowing to emergence, nursery growth and transplanting. The characteristics of this period are decreasing daily maximum temperature, minimum temperature and rising daily relative humidity.

3.2 Grand growth stage: This stage, from July 1st to September 25th includes vegetative and reproductive growth periods. Critical stages of this period are tillering, penicle initiation and flowering.

3.3 Maturity stage: This stage includes the ripening stage of the crop covering from September 25th to October 22nd. Which can be characterized by decreasing maximum temperature, minimum temperature, relative humidity at 07.00 hrs and 14.00 hrs, total rainfall and number of rainy days per week whereas, sun shine hrs and pan evaporation showed increasing trend.

4. Models for studying the effect of weather on crop yield

Several researcher in past developed the forecast models based on weather parameters but most of them were limited to simple correction and regression analysis. However, application of multiple regression technique is better compared to compressive analysis of crop weather relationship. Fisher [5] was the first person who worked on this topic. He handled this problem with the help of polynomial function. hendrick and Schol [6] modified the Fisher's technique and they devided whole crop seasons into *n* weekly intervals and assumed that a second degree polynmial in week number will be sufficiently flexible to express the relationship. The model suggested by Hendrick and Schol has been modified by Agrawal *et al.* [2] by expressing effect of changes in weather variables on yield in w^{th} week as second degree polynomial in correction coefficient between yield an [1] by expressing the effect of change in weather variables on yield in w^{th} week as a linear function of respective correlation between yield weather variables.

In the present study the used model was based on linear term of weather variables and correlation coefficient as weight. The model was as

$$Y = a + \sum_{i=1}^{p}\sum_{j=0}^{1} b_{ij}Z_{ij} + \sum_{i\neq i'=1}^{p}\sum_{j=0}^{1} b_{ii'j}Z_{ii'j}$$

where Z_{ij} and $Z_{ii'j}$ are generated variables given as

$$Z_{ij} = \sum_{w=1}^{m} r_{iw}^{j} x_{iw} \text{ and } Z_{ii'j} = \sum_{w=1}^{m} r_{ii'w}^{j} x_{iw} x_{i'w}$$

and x = weather variable $i\ {}^{1}\ i' = 1,2,3,\ldots\ldots\ldots,p$, p = number of weather variables under study $j = 0, 1$. m = week up to forecast ($m<n$), w = week number (1,2, ……, m), *riw* = correction coefficient between yield (y) and i^{th} weather variable in w^{th} week. *rii'j* = correlation coefficient between yield (y) and product of it^{h} and i^{th} weather variables in w^{th} week *a*, *bij*, *bii'j* are constants. This model was suggested by Agrawal *et al* [3]. The selection and rejection of the generated variables were made using step wise regression analysis suggested by Drapper and Smith [4].

RESULTS AND DISCUSSION

Forecasting of crop yield: The forecast models are obtained from 12 weeks to 20 weeks after sowing. The value of coefficient of multiple determination (R_2) of regression equations varied from 71 to 85 percent. The value of coefficient of multiple determination (R_2) is lowest (i.e., 71%) in 12th and 13th week after sowing and highest (85 percent) in 18th week after sowing. The values of R_2 at different weeks from 12th week to 20th week are as 71%, 71%, 80%, 82%, 80%, 84%, 85%, 83%, and 84% respectively. Keeping in view the earliness of forecasting yield i.e., about 1½ months before harvest, high R_2 value (0.84) and the lower number of generated variables i.e., 6 variables involved in the yield forecasting model is considered appropriate for the forecasting .

Yield forecast model: The yield forecast model obtained for the rice crop at 17th week including six generated variables having 84 percent R_2 is as follows:-

$$Y = 18.63 + 0.0009Z461 + 0.016Z481 + 0.045Z131 - 0.0021Z680 + 0.031Z231 - 1.43Z21,$$

where,

The six weather variables involved in yield forecast model are maximum temperature (X_1), minimum temperature (X_2), relative humidity at 0700 hrs (X_3), relative humidity at 1400 hrs (X_4), total weekly rainfall (X_6) and pan evaporation (X_8).

Simulated forecast yield

In order to study reliability of the developed forecast model the simulated forecast of the rice crop yield based forecast model is made for three subsequent years i.e., 1999-2000, 2000-

yield is to study the reliability of the model.

Table-1. Comparison between actual and simulated yield of rice.

year	Predicted Yield (Q/ha)	Actual (Q/ha)	Deviation (%)
1999-00	42.98	43.05	0.156
2000-01	40.86	390.9	3.94
2001-02	42.87	41.50	3.19

SUMMARY AND CONCLUSION

The forecast models obtained at 12th and 13th weeks are having minimum coefficient of multiple determination (R_2) and models obtained at 18th week has maximum R_2. The value of R_2 obtained at 17th week is 84 percent which is nearer to the value of R_2 (85 percent) obtained at 18 week. Therefore, forecast model obtained at 17th week has been selected for the forecast purpose. The six weather variables involved in the equation are maximum temperature, minimum temperature, relative humidity at 0700 hrs, relative humidity at 1400 hrs, total rainfall per week and pan evaporation. We can easily observe from the table-1 that the deviation between actual and simulated yield is very less i.e., 3.94 percent. Rice crop at Pantnagar can be forecasted at 17th week (2nd week of September) i.e., about one and half month before harvest. Since, Pantnagar farm is an organized farm and the actual yield of the farm is available hence this study may be used as a base for developing the forecast model on large area.

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OPTICAL AND STRUCTURAL PROPERTIES OF BORON DOPED ZNO THIN FILMS PREPARED BY SOL-GEL TECHNIQUE

L.P. Purohit* , Vinod Kumar* and R. Kumar *

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ABSTRACT

Boron doped ZnO films have been prepared on glass substrates by the sol gel spin coating method. Zinc acetate solutions of 0.2 M in methanol stabilized by monoethanolamine and doped with boron tri methyl borate were used. Doping concentration of boron is fixed 0.2 at.%. After deposition samples were air annealed at 450°C for 1 h to improve their optical properties. X- ray diffraction shows that all films are oriented along the c-axis direction of the hexagonal crystal structure. The average optical transmittance of all films is higher than 85%. The effect of boron doping in band gap has been also measured.

Key words :- Zinc oxide, Sol-gel, Boron, Optical properties, Structural properties

INTRODUCTION

Zinc oxide (ZnO) is an II-VI semiconductor, highly transparent (85-95%) in the visible region with a wide and direct band-gap of about 3.3eV [5] at room temperature and a high exciton binding energy of 60 meV. Thin films of undoped and doped ZnO are utilized for a wide variety of electronic and opto-electronic applications, such as transparent conducting electrodes [12], heat mirrors [7] and surface acoustic wave devices [2]. Transparent conducting oxides (TCO) have a range of useful applications such as transparent electrodes in optoelectronic devices and solar cells.

Boron doped ZnO thin films were prepared by metal organic chemical vapor deposition [8], sputtering [3,6,9] and laser deposition [11]. However, boron doped ZnO thin films prepared by sol-gel method [13] are rare. The sol-gel method offers many advantages such as highly homogeneous thin films, large area coating, and absence of the need for vacuum, low cost and high flexibility.

In this paper structural and optical properties of sol-gel derived boron doped ZnO films using spin coating techniques has been investigated.

EXPERIMENTAL DETAILS

Boron doped ZnO (ZnO: B) thin films were prepared by sol gel method using spin coating techniques. The starting material for boron doped ZnO thin films was zinc acetate dihydrate [Zn $(CH_3COO)_2 \cdot 2H_2O$]. The dopant source of boron was Tri methyl borate (99.999%, high purity chemicals). Methanol was used as an alcoholic medium in this processing. Monoethanolamine

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(MEA) is used as a complexing agent to keep the metal ions in homogeneous solution.

The Zn precursor solution was prepared by dissolving zinc acetate dihydrate in methanol so as to prepare concentration of 0.2 mol/L. The molar ratio MEA/Zn was fixed to 1. Finally, trimethyl borate was added to provide a stock solution with specific molar ratio of boron to zinc 0.2 at.%. The solution was dropped onto corning glass (1737) substrates, which were rotated at 2500 rpm for 30 s. After each deposition, the resulting film was then dried in air at 230°C for 15 min in an oven in order to remove the solvents and organic compounds by evaporation. The boron doped films were annealed in air at 450°C for one hour. The flow chart for the processing of ZnO:B thin films is shown in Fig.1. Crystallinities of the ZnO: B films were confirmed by x-ray diffractometer (PAN alytical X'pert PRO) using CuK$_\alpha$ radiation (λ = 1.5140Å). Optical transmittance measurements were carried out using UV-VIS-NIR spectrophotometer (Shimadzu Solid Spec 3700).

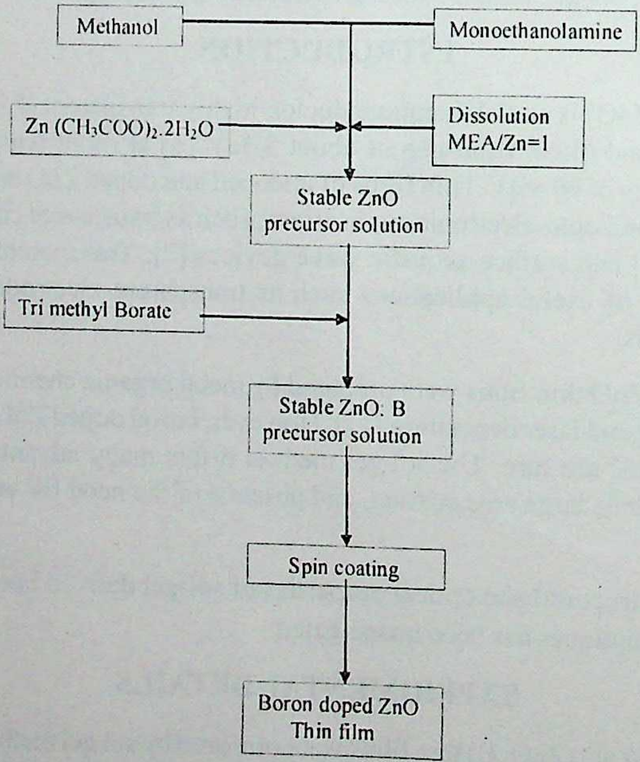
Methanol
Monoethanolamine
Zn (CH3COO)2.2H2O
Dissolution
MEA/Zn=1
Stable ZnO
precursor solution
Tri methyl Borate
Stable ZnO: B
precursor solution
Spin coating
Boron doped ZnO
Thin film

Fig.1 Flow chart for Born doped ZnO films prepared by sol- gel method

RESULTS AND DISCUSSION

Structural Properties

The X-ray diffraction spectra of boron doped ZnO films deposited by sol gel techniques at 0.2 at.% boron doping concentration is shown in Fig. 2. The X-ray spectra of undoped ZnO prepared by sol-gel are also inducted in the figure for the comparison. The registered X-ray spectra fit to a ZnO hexagonal wurtzite structure without any new reflections. No additional peaks of ZnO phases and the traces of either B or Zn metals have been observed. This indicates that the B^{+3} are substituted at Zn^{+2} sites without changing the ZnO structure.

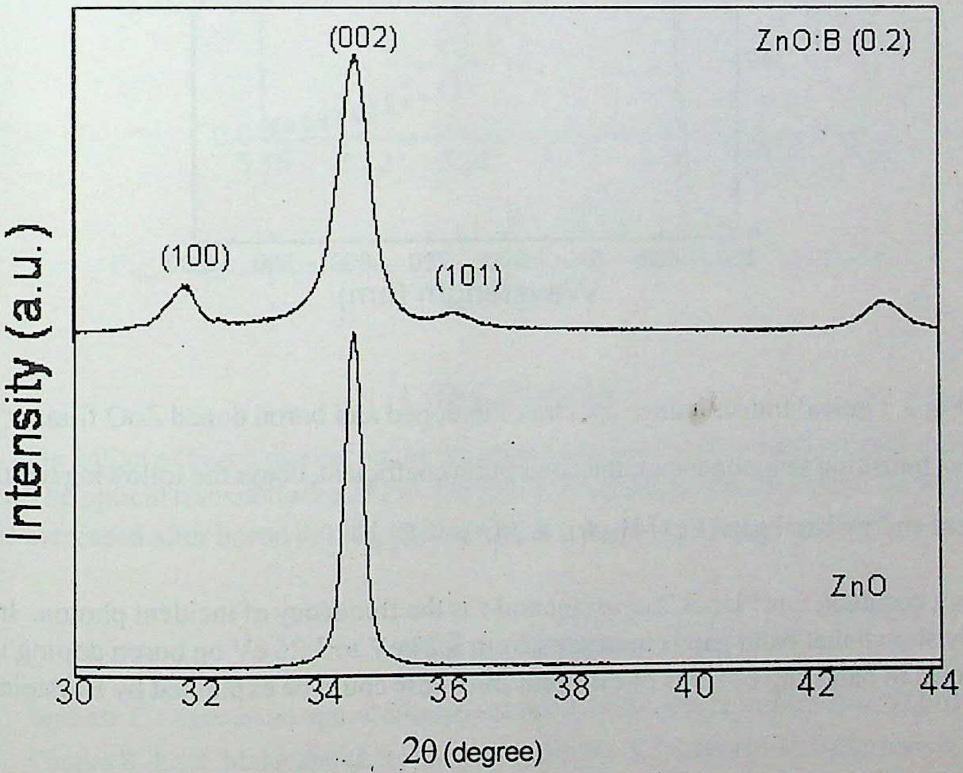

Fig.2. XRD Spectra of undoped and boron doped ZnO films.

Optical Properties

The optical transmission spectra of the film deposited on corning glass substrates taken in the wavelength region 300-1000 nm are shown in Fig 3. The maximum transmittance in visible region was found to be 87%. This value is significantly higher than those of most transparent conductive oxides. The transmission data were used to evaluate absorption coefficients of the boron doped thin films at different wavelengths. At the absorption edge, the absorption coefficient

(α) is given by the following relation [10]

$$T \approx \exp^{(-\alpha d)}$$

where d is film thickness and α is absorption coefficient.

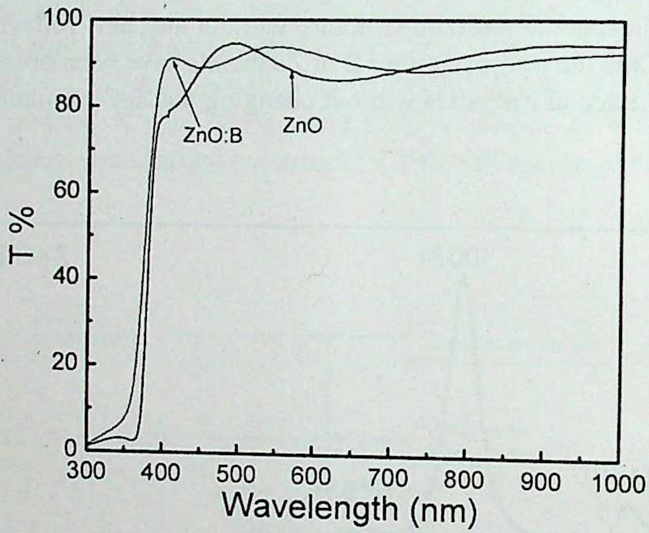

Fig.3. Optical transmittance spectra of undoped and boron doped ZnO films.

In the direct transition semiconductor, the absorption coefficient, obeys the following relationship with optical energy band gap (Eg) [4] $\alpha h\nu = B\left(h\nu - E_g\right)^{1/2}$

where B is a constant, h is Planck's constant and ν is the frequency of incident photon. In Fig.4, it is clearly shown that band gap is increased from 3.24 eV to 3.26 eV on boron doping in ZnO. The variation in band gap of films of different thickness could be explained by Burstein–Moss shift [1].

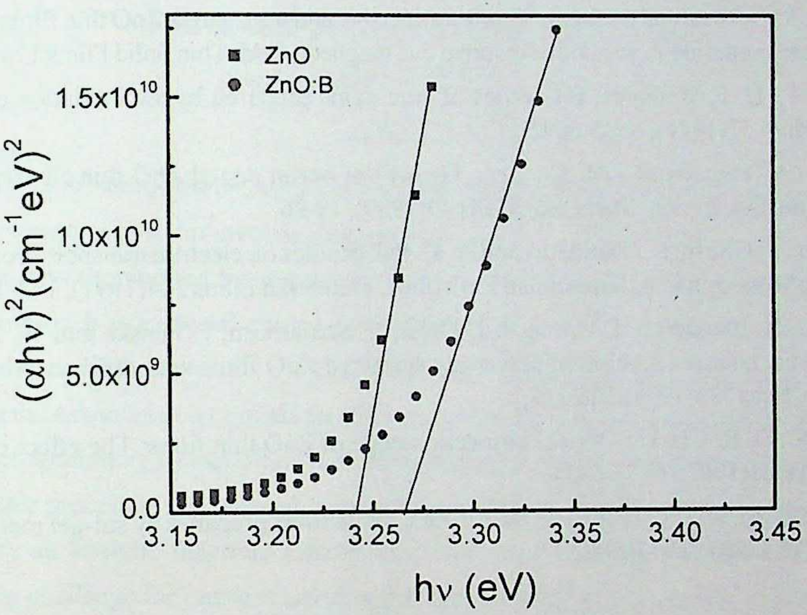

Fig.4. Optical band gap of undoped and boron doped ZnO films.

CONCLUSION

The effect of boron doping on the structural and optical properties of ZnO thin films was studied. The optical transmittance of ZnO: B film is higher than the ZnO film. Band gap of Zinc oxide is increased after boron doping in Zinc Oxide.

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ON THE FOUNDATION OF THE ADOMIAN DECOMPOSITION METHOD

Fawzi Abdelwahid [*] and Randolph Rach[**]

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ABSTRACT

In this work, we review the Adomian decomposition method for the case of a nonlinear differential equation with a general operator involving a special parameter λ., and examine the Taylor series expansion of a nonlinear term represented by an analytic function. For this type of operator, this study shows that Adomian's approach is a special choice corresponding to $\lambda=1$. This study also defines the Adomian polynomials and explains Adomian's choices. As a result of this study, we feature Abdelwahid's Algorithm for generating the Adomian polynomials for different types of nonlinear terms, which is eminently suitable for symbolic programming by exploiting the sifting property of the well-known Kronecker delta function. Furthermore this procedure can be used to calculate the Adomian polynomials for any nonlinear term as represented by an analytic function. Finally this study indicates that the case of $\lambda \neq 1$ presents an interesting new challenge for future research and studies including the case of $\lambda = \lambda\,(x)$ and introduces a general approach for the Adomian decomposition method.

Keywords: Adomian decomposition method; Adomian polynomials; Adomian decomposition series; Adomian's Integral Equation; parameterized Taylor expansion series; analytic parameter, nonlinear deterministic differential equations

INTRODUCTION

Adomian decomposition method [1-7] is a significant, powerful method, which provides an efficient means for the analytic and numerical solution of differential equations, which model real-world physical applications. In recent years Adomian and others have successfully applied the decomposition method to ordinary, partial, delay [8-9], and non-integer order [10] differential equations for a wide class of nonlinearities, including polynomial, exponential, trigonometric, composite, negative power [11], radical [12] and even decimal power [13] nonlinearities. Adomian's decomposition method allows us to solve nonlinear differential equations without having to appeal to the decidedly questionable practices of perturbation or linearization. Furthermore Adomian et al. [14-15] have developed a new approach to numerical integration algorithms based upon Adomian's decomposition method.

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One of our aims in this paper is to explain the basis of the Adomian decemposition method and its widespread applicability. To achieve this aim, we will review Adomian's approach for a nonlinear ordinary differential equation in the form of Adomian Differential Equation

$$Fu = Lu + Ru + Nu = g \tag{1.1}$$

Here g is a specified analytic function and F represents a general nonlinear stochastic operator; L is the linear operator, R is the random or stochastic operator and N is the nonlinear operator. In this study, we consider the general operator F to be a function of a special parameter $\lambda = 0$. For simplicity we may assume that equation (1.1) corresponds to an initial value problem with a nonlinear deterministic ordinary differential equation. In general our approach can be used for boundary value problems [16-19] as well as other types of operators including stochastic differential equations, integral equations, non-integer order differential equations, etc. We also aim to define the Adomian polynomials and explain Adomian's choices. To achieve these goals, we will study the Taylor series expansions of a nonlinear term as represented by an analytic function.

ADOMIAN'S APPROACH

We consider natural and engineering systems as modeled by the deterministic form of equation (1.1) or equivalently

$$Lu + Nu = g, \tag{2.1}$$

where generally, the operator F has linear and nonlinear components. Adomian and coworkers partition L into components for sake of easy integrations, i.e. trivial Green functions, such that $L = L + R$, where, $L = d^p/dx^p$ is the highest order linear differential operator and R is the remainder operator of linear terms, which has the general form $R = \sum_{\nu=0}^{p-1} \alpha_\nu(x) \frac{d^\nu}{dx^\nu}$ We accept a trade-off, where the price of trivial integrations is a deceleration in a prospective rate of convergence. For purposes of illustration, we specialize the nonlinear operator N to be a simple nonlinearity such as $R_2 f(\mathrm{u})$, where $f(u)$ is an analytic function of the solution u and where, in this case, the operator R_2 may have a similar form as the remainder operator R_1. More generally, for example, N may include the general form $f\left(u, \frac{du}{dx}, \ldots\ldots, \frac{d^{p-1}u}{dx^{p-1}}\right)$ however then R_2 could not include derivatives, etc. We restrict our discussion further to the case of ordinary differential equations for simplicity of exposition. Hence we write the illustrative case

$$Lu - R_1 u - R_2 f(u) = g\ . \tag{2.2}$$

We observe that this is in the usual form of Adomian and coworkers except for the replacement of $R = - R_1$ and $Nu = - R_2 f(u)$, which does not affect our derivation except to suppress powers of negative one, and is adopted for the sake of clarity. We remark that this choice of L is easily invertible as $L^{-1} = F + A^{-1}$, where A^{-1} is understood to be *a* p-fold indefinite integral operator $\int \cdots \int (.)\, dx....dx$ and Φ comprises the constants of integration such that $L\,\Phi = 0$, hence $\Phi = \sum_{v=0}^{p-1} \beta_v \frac{x^v}{v!}$. The coefficients β_v are the matching coefficients for satisfying either initial conditions or boundary conditions. We remark that customarily the initial conditions or boundary conditions are incorporated into the respective Green's function. Φ is re-evaluated at each stage of solution approximation; this allows us to use the same definition for the inverse operator L^{-1} for either initial value problems or boundary value problems. Previously Adomian and coworkers have employed the definition $L^{-1} = \int_{0.}^{t} .....\int_{0.}^{t} (.)dt......dt$ for initial value problems, i.e. they interpreted L^{-1} as a p-fold definite integral operator when initial conditions are specified. In the case of the inverse linear operator L^{-1} for the case of second-order differential equations with singular coefficients, we can choose $L^{-1} = \int_{.0}^{r} r^{-2} \int_{0}^{r} . \int_{.0}^{r} r^2 (.)dr\,dr$, when $L = r^{-2} \frac{d}{dr}(r^2 \frac{d}{dr})$, or more generally $L^{-1} = \beta_0 + \beta_1 r + \int r^{-2} \int r^2 (.)drdr$, for either initial value problems or boundary value problems, and so forth for higher order differential equations. We emphasize that the linear operator L and thus its inverse L^{-1} is determined by the particular equation to be solved. We reserve the freedom to choose how to decompose the operators that compose Adomian's Differential Equation (1.1) for sake of computational convenience [20], i.e. $Fu + Lu + Ru + Nu = g$, Often our examples in discourse are linear and deterministic or nonlinear and deterministic. This has been for the sake of clarity in exposition, and does not reflect any restrictions. The Adomian decomposition method was created and developed for solving faithful mathematical models that characterize physical systems, which can and often do include nonlinearities and general stochastic processes.

Next we solve for the solution *u* as modeled by

$$Lu = g + Rju + R_2 f(u) \tag{2.2}$$

We require the Banach theorem because the coefficient operators of the type $L^{-1}R$, specifically $L^{-1}R_1$ and $L^{-1}R_2$, need to be contractive, i.e. their norms are less than one: $\left\|L^{-1}R_1\right\| < 1$ and $\left\|L^{-1}R_2\right\| < 1$. This is a practical prerequisite because in the sequel the solution is constructed in terms of powers of these $L^{-1}R_1$ and $L^{-1}R_2$ coefficient operators; Adomian's decomposition method is after all a constructive scheme. Upon inverting the linear operator *L*, we have

$$L^{-1}Lu = L^{-1}g + L^{-1}R_1 u + L^{-1}R_2 f(u) \tag{2.3}$$

which yields Adomian's Integral Equation

$$u = \Phi + L^{-1}g + L^{-1}R_1 u + L^{-1}R_2 f(u) \tag{2.4}$$

We employ the Picard-Lindelöf theorem of existence and uniqueness for ordinary differential equations and the Cauchy-Kovalevskaya theorem of existence and uniqueness for partial differential equations. Essentially these theorems guarantee analytic solutions *u* for the prerequisites of analytic input functions *g*, bounded initial and boundary conditions, which are incorporated into the homogeneous term Φ, analytic coefficients $a_v(x)$, which are incorporated into the remainder operators R_i, and analytic nonlinear functions $f(u)$ in the solution. Inexplaining his approach, Adomian has utilized the notion of analytic parameterization to craft the Adomian decomposition series

$$u = \sum_{m=0}^{\infty} u_m \lambda^m \text{ and } f(u) = \sum_{m=0}^{\infty} A_m \lambda^m \tag{2.5}$$

We remark that to approach the basis of the Adomian Decomposition Method, we further apply the Murray- Miller theorem [21] of existence and uniqueness, which requires that the function *u* and the composite function $f(u)$ are analytic in a parameter λ, or parameters λ_i. Thus we take advantage of the parameterized Taylor series expansion of the analytic functions $u = u(x;\lambda)$ and $f(u) = f(u(x;\lambda))$

$$u = \sum_{m=0}^{\infty} u_m(x)\lambda^m \text{ and } f(u) = \sum_{m=0}^{\infty} A_m(x)\lambda^m \tag{2.6}$$

Since the parameter λ is a constant, then notice the implied dependence of the components U_m and A_m on the independent variable x, which we have explicitly denoted. For Adomian's choice for the value of λ, these parameterized Taylor expansion series become the Adomian decomposition series. Here we have made the important connection between the Adomian decomposition series and a parameterized version of the Taylor expansion series. Adomian's decomposition method may at first glance appear simplistic on the surface, however Adomian was the first to grasp its strengths and widespread applicability. The better that we understand a method, then the simpler will become our explanations! We concern ourselves only with physical systems and not with pathological cases. Hence the components $u_m(x)$, which compose the solution $u(x)$, and the components $A_m(x)$, which compose the nonlinear function $f(u(x))$, are determined in the usual way by

$$u_m = \frac{1}{m!}\frac{\partial^m u(x;\lambda)}{\partial\lambda^m}\bigg|_{\lambda=0} \qquad A_m = \frac{1}{m!}\frac{\partial^m f(u(x;\lambda))}{\partial\lambda^m}\bigg|_{\lambda=0} \tag{2.7}$$

This becomes the foundation of analytic parameterization, where A is deemed the analytic parameter by Adomian and coworkers. We remark that Murray and Miller were concerned with parameterized solutions of differential equations as solved by electronic analog computers, alias electronic differential analyzers. They sought to rigorously justify their continuous, graphical solutions from their era's analog computers in contrast to discrete solutions from today's digital computers. Their research presents an alternative starting point for approaching the foundation of Adomian's Decomposition Method from the perspective of much hindsight, as we are usually concerned with continuous analytic solutions.

Upon substitution of the two analytic functions with their respective uniformly convergent series, we develop the parameterized decomposition of the nonlinear deterministic differential equation

$$\sum_{m=0}^{\infty} u_m\lambda^m = \Phi + L^{-1}g + L^{-1}R_1\sum_{m=0}^{\infty} u_m\lambda^m + L^{-1}R_2\sum_{m=0}^{\infty} A_m\lambda^m \tag{2.8}$$

Adomian has adapted the analytic parameter A, also used earlier by Murray and Miller, to bea grouping parameter by setting λ = 1. It becomes a simple way to group terms in a manner that yields very convenient forms for computation and lecture. We remark that other values of the analytic parameter λ are possible as in the work of Murray and Miller and as discussed by Adomian and Rach [22], who have considered the generalization of the analytic parameter λ to be a function in the independent variable as $\lambda = \lambda(x)$. Thus we have transformed the Adomian decomposition series of equation (2.6) to

$$u(\lambda(x)) = \sum_{m=0}^{\infty} u_m \lambda^m(x) \text{ and } f(u(\lambda(x))) = \sum_{m=0}^{\infty} A_m \lambda^m(x) \tag{2.9}$$

Next they observed that various functions for $\lambda(x)$ would subsume well-known series as special cases of the Adomian decomposition series of equation (2.6). If λ is chosen to be a very small constant, e.g. $|\lambda| << 1$, we develop a perturbation series or equivalently we may set $\lambda(x) = \varepsilon$, the perturbation parameter, in this special case. If $\lambda(x) = x$, then we have the Maclaurin series, where the components u_m become constants and the A_m also become constants evaluated in terms of the components u_m such that $A_m = A_m(u_0, \ldots, u_m)$ using the formulas for the Adomian polynomials. For appropriate choices of $\lambda(x)$ we can develop a series of orthogonal polynomials, e.g. the Legendre polynomials by applying the Schmidt orthogonalization procedure over an appropriate range, etc. Also consider the choice of $\lambda(x) = \cos(x) + i\sin(x)$, or $\lambda(x) = e^{ix}$, which subsumes the well-known Fourier series. Adomian's choice is thus a special case corresponding to $\lambda = 1$, where the analytic parameter becomes a grouping parameter. We suggest that in-depth investigation for the general case of $\lambda(x) \neq 1$, presents a fascinating challenge for future research and studies.

Next we compute the initial solution component u_0 such that

$$u_0 = \Phi_0 + \Lambda^{-1} g \tag{2.10}$$

where the initial component of the homogeneous term Φ_0 combines the matching coefficients for both the $L^1 L_u$ and $L^1 g$ integrals, and for the index $m \geq 0$ we compute the solution components u_{m+1} such that

$$u_{m+1} = L^{-1} R_1 u_m + L^{-1} R_2 A_m \text{ or } u_{m+1} = \Phi_{m+1} + \Lambda^{-1} R_1 u_m + \Lambda^{-1} R_2 A_m \tag{2.11}$$

where each Φ_{m+1} combines the matching coefficients for both the terms $L^{-1} R_1 u_m$ and $L^{-1} R_2 A_m$ integrals. Practically then we may redefine the inverse linear operator L^{-1} as

$$L^{-1} = \Phi_{m+1} + \Lambda^{-1} \text{ with } \Phi = \sum_{m=0}^{\infty} \Phi_m \lambda^m \tag{2.12}$$

thus we have $\Phi_m = \sum_{\nu=0}^{m-1} \beta_\nu^{(m)} \frac{x^\nu}{\nu!}$ with $\beta_\nu = \sum_{m=0}^{\infty} \beta_\nu^{(m)}$. We remark that the convergence of the series for the homogeneous term $\Phi = \sum_{m=0}^{\infty} \Phi_m$ and the convergence of the series for the coefficiients $\beta_v = \sum_{m=0}^{\infty} \beta_v^{(m)}$ naturally depend upon the convergence of the analytic functions u and $f(u)$, which has been proven elsewhere.

With the Adomian decomposition scheme or recurrence relations

$$u_0 = \gamma \text{ and } u_{m+1} = \Phi_{m+1} + \Lambda^{-1} R_1 u_m + \Lambda^{-1} R_2 A_m \qquad (2.13)$$

where $\gamma = \Phi_0 + \Lambda^{-1} g$, then we compose the solution as represented by the Adomian decomposition series by adding these components

$$u = \sum_{n=0}^{\infty} (L^{-1} R_1)^n \gamma + \sum_{m=0}^{\infty} (L^{-1} R_1)^{n-1-m} L^{-1} R_2 A_m \qquad (2.14)$$

where $\gamma = \Phi_0 + \Lambda^{-1} g, L^{-1} = \Phi_{m+1} + \Lambda^{-1}$, and where the A_m can be any of the four known classes of the Adomian polynomials. Usually we choose to employ the ordinary or classic Adomian polynomials due their sufficiently rapid rate of convergence. Note that this solution in equation (2.14) subsumes the solution as presented by Adomian [3] for a particular case for the replacement of R_2 by -1 and R_1 by $-R$. Of course this formula of the Adomian decomposition series solution subsumes the special cases for either the case of only linear terms when $R_2 = 0$

$$u = \sum_{n=0}^{\infty} (L^{-1} R_1)^n \gamma \qquad (2.15)$$

or the case of only nonlinear terms when $R_1 = 0$

$$u = \gamma + \sum_{n=0}^{\infty} L^{-1} R_2 A_n \qquad (2.16)$$

We observe from this expression of Adomian decomposition series solution in equation (2.14) that the coefficient operators $L^{-1} R_i$ must be contractive while the nonlinear function $f(u)$ need only be analytic as proven elsewhere.

The Adomian Decomposition Method in essence effects a double decomposition of the nonlinear composite function $f(u)$ by substituting the decomposition series of the sol ution u into the Banach-space analog of the Taylor expansion series about the function $u_0(x)$, rather than about a constant such as the usual x_0. Thus

$$f(u(\lambda)) = \sum_{m=0}^{\infty} \frac{1}{m!} (\sum_{n=0}^{\infty} u_n \lambda^n - u_0)^m f^{(m)}(u_0) \tag{2.17}$$

$$f(u(x;\lambda)) = \sum_{m=0}^{\infty} \frac{1}{m!} (\sum_{n=0}^{\infty} u_n(x) \lambda^n - u_0(x))^m f^{(m)}(u_0(x))$$

where $f^{(m)}(u_0) = \frac{\partial^m}{\partial u^m} f(u_0)$. Then upon subtraction of the initial solution component u_0 we have found

$$f(u(\lambda)) = \sum_{m=0}^{\infty} \frac{1}{m!} (\sum_{n=0}^{\infty} u_n \lambda^n)^m f^{(m)}(u_0) \tag{2.18}$$

The initial solution component u_0 comprises all a priori system information as Adomian's choice $u_0 = \Phi_0 + \Lambda^{-1} g$. Other choices for u_0 are possible. We remark that Wazwaz [23 - 24] has adopted a different decomposition of u_o in order to obviate the appearance of so-called noise terms occurring in special cases [25 - 26], thus precluding the phenomenon of damped oscillating convergence in such a circumstance. Again we find that the decomposition is not unique. This double decomposition of the nonlinear compositefunction $f(u)$ yields another formula for the series of the Adomian polynomials. We compute the m-th order Adomian polynomial A_m by equating the two formulas for the analytic composite function $f(u)$ thus

$$\sum_{m=0}^{\infty} A_m \lambda_m = \sum_{n=0}^{\infty} \frac{1}{m} (\sum_{n=1}^{\infty} u_n \lambda^n)^m f^{(m)}(u_0) \tag{2.19}$$

then next we equate terms in like powers of the analytic parameter λ.

Since Cherruault et al. have rigorously established convergence of the Adomian decomposition series in several papers [27 - 30], then let us consider the basis for the accuracy and rapid rate of convergence of the Adomian Decomposition Method. The initial component u_0 is an optimal first approximation containing essentially all a priori information about the system. Thus $u_0 = \Phi_0 + \Lambda^{-1} g$ contains the contribution from the system input g, which is bounded

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in a physical system, and the initial or boundary conditions are incorporated into Φ. The definition of the linear operator L avoids difficult or cumbersome integrations involving nontrivial Green's functions. The use of a finite approximation in series form for the system input g, and calculation to only the necessary accuracy simplifies integrations still further. The mode of convergence for the Adomian decomposition series $u = \sum_{n=0}^{\infty} u_n$ and the series of Adomian polynomials $f(u) = \sum_{n=0}^{\infty} A_n$ has been shown to be uniformly convergent from the Murray-Miller theorem for existence and uniqueness, i.e. the Adomian decomposition series are uniformly convergent because they are essentially a rearrangement of the parameterized Taylor expansion series for analytic functions. Furthermore, for bounded x and bounded input $g(x)$, the Adomian decomposition series is characterized by a rapid rate of convergence as the m-th order solution component u_m is found to be proportional to $\frac{x^{mp}}{(mp)!}$, where p is the order of the differential equation, and hence, of course, the linear operator L, and m is the number of terms in the solution approximation $\phi_m = \sum_{n=0}^{m-1} u_n$. Hence with even very small index m, the solution approximants, i.e. the partial sums ϕ_m, will contain most of the solution u. In practice this means that few terms are required as shown in numerous examples and applications. Further Adomian and Rach [31] have demonstrated that $f(u)$ need not even be analytic but could be a piecewise-differentiable function by representation as a series of analytic functions. Since the Adomian decomposition series $u = \sum_{n=0}^{m-1} u_n$ uniformly converges very rapidly, then the m-term approximation, i.e. partial sum, $\phi_m = \sum_{n=0}^{m-1} u_n$ serves as a practical solution for purposes of synthesis and design. We remark that this derivation rigorously justifies Adomian's previous usage of the several properties of uniformly convergent series of analytic functions in his solution methodology, i.e. term-by-term integration, term-by-term differentiation, associative rearrangement of summands within sums, and round-off or uniform approximation over the region of convergence, in all of his work.

To continue our computation of solution components further than the initial solution component u_0, we shall require formulas for the Adomian polynomials A_m. Earlier Adomian and coworkers determined the Adomian polynomials by implicit differentiations

$$A_m = \frac{1}{m!}\frac{\partial^m f(u(x;\lambda))}{\partial\lambda^m}\bigg|_{\lambda=0} \tag{2.20}$$

The Adomian polynomials depend only on the components from u_0 to u_m i.e. $A_m = A_m(u_0,\dots,u_m)$. We succinctly state the definition of the Adomian decomposition series: An Adomian decomposition series of an analytic function u or composite analytic function $f(u)$ is a rearrangement of its uniformly convergent Banach-space analog Taylor expansion series about the analytic function u_0, the initial solution component, which permits computation of solution components u_m by recursion. We designate the Adomian decomposition series for the composite analytic function $f(u)$ to be the series of the Adomian polynomials $\sum_{m=0}^{\infty} A_m$. In other words the Adomian decomposition series, including the seriesof Adomian polynomials, is the result of a uniformly convergent series substituted into another uniformly convergent series.

In 1983 Adomian and Rach [32] presented the formula for the Adomian polynomials for the general case of analytic functions $f(u)$, where, for $m \geq 1$, the m-th Adomian Polynomial A_m is

$$A_m = \sum_{v=1}^{m} c(v,m)\frac{d^v f}{du^v}\bigg|_{\lambda=0} \tag{2.21}$$

where the coefficients $c(v,m)$ for $1 \leq v \leq m$ are calculable from the recurrence relation

$$c(i,j) = \frac{d}{d\lambda}\{c(i,j-1)\} + \left(\frac{du}{d\lambda}\right)\{c(i-1,j-1)\} \tag{2.22}$$

for $L \leq i, j \leq m$, where $c(0,0) = 1$ and $c(1,0) = 0$ from inspection. Thus the series of Adomian polynomials compose the nonlinear function $f(u)$ such that

$$f(u)=\sum_{m=0}^{\infty}A_m=\sum_{m=0}^{\infty}\ \sum_{v=1}^{m}c(v,m)f^{(v)}(u_0). \tag{2.23}$$

Later Rach [33] presented the Symmetry Rule for Adomian Polynomials comprised in the formula for the coefficients $c\,(v\,,m)$ such that

$$c(v,m)=\sum_{\sum_{i=1}^{v}k_i r_i=m}\ \prod_{i=1}^{v}\frac{u_{r_i}^{k_i}}{k_i\,!} \tag{2.24}$$

where k_i is the number of repetitions in the solution components u_{r_i}, $m>0$, $1\le i\le m$, the values of the indices $1\le r_i\le(m-v+1)$, and the values of r_i are seleted from this range by combinations without repetition.

ADOMIAN'S POLYNOMIALS

The Adomian polynomials represent the main factor of Adomian's decomposition method in solving nonlinear differential equations, and we now follow yet a different path leading to a convenient computer algorithm of the Adomian polynomials. In this section, we aim to define the Adomian polynomials and explain Adomian's choice. To achieve these goals, we will investigate the Taylor expansion series for the nonlinear term represented by an analytic function, say $f(u)$. First we develop the Taylor expansion series of

$f(u)$ about the zeroth component u_0 as

$$f(u)=\sum_{n=0}^{\infty}\left(\sum_{i=1}^{\infty}u_i\lambda^i\right)^n\frac{f^{(n)}(u_0)}{n!}, \tag{3.1}$$

which, following the approach of Abdelwahid [34] utilizing the Kronecker delta function, can be written as

$$f(u)=f(u_0)+\sum_{n=1}^{\infty}\lambda^n\{\sum_{v=1}^{n}\left(\frac{1}{v!}\sum_{i_1,i_2,\ldots,i_v=1}^{n+1-v}\delta_{n,i_1+i_2+\ldots+i_v}u_{i_1}u_{i_2}\cdots u_{i_v}\right)f^{(v)}(u_0)\} \tag{3.2}$$

where $D_{n,m}$ is the well-known Kronecker delta function and we have exploited its sifting property to our advantage in the sequel. To define the Adomian polynomials A_n, we equate the Taylor expansion series in the form of equation (3.2) to the Adomian decomposition series of the nonlinear function $f(u)$ in the form of equation (2.6)

$$\sum_{n=0}^{\infty} \lambda^n A_n = f(u_0) + \sum_{n=1}^{\infty} \lambda^n \{ \sum_{\nu=1}^{n} \left(\frac{1}{\nu!} \sum_{i_1, i_2, \dots, i_\nu = 1}^{n+1-\nu} \delta_{n, i_1+i_2+\dots+i_\nu} u_{i_1} u_{i_2} \cdots u_{i_\nu} \right) f^{(\nu)}(u_0) \} \quad (3.3)$$

then compare the coefficients of like powers of λ on both sides of equation (3.3). Accordingly we have developed Abdelwahid's Algorithm for generation of the Adomian polynomials

$$A_0 = f(u_0)$$

$$A_n = \sum_{\nu=1}^{n} \left(\frac{1}{\nu!} \sum_{i_1, i_2, \dots, i_\nu = 1}^{n+1-\nu} \delta_{n, i_1+i_2+\dots+i_\nu} u_{i_1} u_{i_2} \cdots u_{i_\nu} \right) f^{(\nu)}(u_0) \text{ for } n = 1,2,3\dots\dots \quad (3.4)$$

This leads to the understanding of Adomian's approach to decomposing nonlinear functions in terms of the solution components as $A_n = A_n(u_0, \dots, u_n)$ that permits resolution of the solution of nonlinear differential equations by recursion, and explains the choice of Adomian's polynomials. This defines the Adomian polynomials in an apt sense for symbolic programming languages such as the widely available MATHEMATICA, as well as offers formulas for the coefficients $c(\nu, n)$ in equation (2.22) thus

$$c(\nu,n) = \frac{1}{\nu!} \sum_{i_1, i_2, \dots, i_\nu = 1}^{n+1-\nu} \delta_{n, i_1+i_2+\dots+i_\nu} u_{i_1} u_{i_2} \cdots u_{i_\nu} \quad for\, n \geq \nu \quad (3.5)$$

where, of course, $\delta_{n,m}$ is the Kronecker delta function.

This study also introduces a general procedure for generating the Adomian polynomials for any type of nonlinear terms, depending on our assumption that the nonlinear term is an analytic function. As an application of this study, we calculate the Adomian polynomials for chosen nonlinear terms.

(I) For the quadratic nonlinearity $N_u = u^2$, we make use of the Adomian decomposition series $u = \sum_{=0}^{\infty} \lambda^n u_n$ for the solution to determine the resulting Cauchy product

$$u^2 = (\sum_{n=0}^{\infty} \lambda^n u_n)^2 = \lambda^0 (u_0^2) + \lambda (2u_0 u_1) + \lambda^2 (u_1^2 + 2u_0 u_2) + \lambda^3 (2u_1 u_2 + 2u_0 u_3) + \dots \quad (3.6)$$

Next we equate this expression with the Adomian decomposition series $u = \sum_{n=0}^{\infty} \lambda_n A_n$ and compare like powers of λ on both sides. This yields

$$\begin{aligned} A_0 &= u_0^2 \\ A_1 &= 2u_0 u_2 \\ A_2 &= u_1^2 + 2u_0 u_1 \\ A_3 &= 2u_1 u_2 + 2u_0 u_3 \end{aligned} \tag{3.7}$$

$$A_n = \sum_{\nu=1}^{n} \left(\frac{1}{\nu!} \sum_{i_1, i_2 \ldots\ldots i_\nu = 1}^{n+1-\nu} \delta_{n, i_1 + i_2 + \ldots + i_\nu} u_{i_1} u_{i_2} \ldots\ldots\ldots u_{i_\nu}\right) \frac{\partial^\nu}{\partial u_0^\nu}(u_0^2)\; for\, n = 1,2,3$$

(II.) For a product nonlinearity $Nu = u^2 u_x$, where $u_x = \frac{\partial u}{\partial x}$ we make use of the Adomian decomposition series $u = \sum_{n=0}^{\infty} \lambda^n u_n$ for the solution and perform term-by-term differentiation

$$u = \sum_{n=0}^{\infty} \lambda^n u_{n,x} \tag{3.8}$$

Next we multiply these two infinite series to determine their Cauchy product

$$u^2 u_x = \left(\sum_{n=0}^{\infty} \lambda^n u_n\right)^2 \left(\sum_{n=0}^{\infty} \lambda^n u_{n,x}\right) \tag{3.9}$$

This gives

$$\begin{aligned} u^2 u_x = {}& \lambda^0 (u_0^2 u_{0,x}) + \lambda^1 (2u_0 u_1 u_{0.x} + u_0^2 u_{1.x}) + \lambda^2 (2u_0 u_2 u_{0.x} + u_1^2 u_{0.x} + 2u_0 u_1 u_{2.x} + u_0^2 u_{i.x}) \\ & + \lambda^3 (2u_1 u_2 u_{0.x} + 2u_0 u_3 u_{0.x} + u_1^2 u_{1.x} + 2u_0 u_2 u_{1.x} + 2u_0 u_1 u_{2.x} + u_0^2 u_{3.x}) + \ldots\ldots \end{aligned} \tag{3.10}$$

Next we equate this expression with the Adomian decomposition series $u = \sum_{n=0}^{\infty} \lambda^n A_n$ and powers of λ on both sides. This yields (recalling that $u_x = \dfrac{\partial u}{\partial x}$ in the following calculations)

$$\begin{aligned}
A_0 &= u_0^2 u_{0,x} \\
A_1 &= 2u_0 u_1 u_{0,x} + u_0^2 u_{1,x} \\
A_2 &= 2u_0 u_2 u_{0,x} + u_1^2 u_{0,x} + 2u_0 u_1 u_{2,x} + u_0^2 u_{1,x} \\
A_3 &= 2u_1 u_2 u_{0,x} + 2u_0 u_3 u_{0,x} + u_1^2 u_{1,x} + 2u_0 u_2 u_{1,x} + 2u_0 u_1 u_{2,x} + u_0^2 u_{3,x} \\
&\vdots
\end{aligned} \tag{3.11}$$

$$A_n = \sum_{\nu=1}^{n} \left(\frac{1}{\nu!}\right)\left(\frac{1}{\nu!}\sum_{i_1,i_2,\ldots,i_\nu=1}^{n+1-\nu} \delta_{n,i_1+i_2+\ldots+i_\nu} u_{i_1} u_{i_2} \cdots\cdots u_{i_\nu}\right)\frac{\partial^\nu}{\partial u_0^\nu}(u_0^2 u_{0,x}) \text{ for } n = 1,2,3,\ldots.$$

(III.) For a trigonometric non-linearity $Nu = \sin(u)$, we make use of the definition in equation (3.4). This yields

$$\begin{aligned}
A_0 &= \sin(u_0) \\
A_1 &= u_1 \cos(u_0) \\
A_2 &= -\frac{u_1^2}{6}\sin(u_0) + u_2 \cos(u_0) \\
A_3 &= -\frac{u_1^3}{6}\cos(u_0) - u_1 u_2 \sin(u_0) + u_3 \cos(u_0) \\
&\vdots
\end{aligned} \tag{3.12}$$

$$A_n = \sum_{\nu=1}^{n} \left(\frac{1}{\nu!}\sum_{i_1,i_2,\ldots,i_\nu=1}^{n+1-\nu} \delta_{n,i_1+i_2+\ldots+i_\nu} u_{i_1} u_{i_2} \cdots\cdots u_{i_\nu}\right)\frac{\partial^\nu}{\partial u_0^\nu}\sin(u_0) \; for\, n = 1,2,3,\ldots$$

We have used the MATHEMATICA software package to verify these results, cf. the MATHEMA TICA program in Abdelwahid [34]; for example, the reader can compare these results with the computation in Wazwaz [35]. In general, this procedure can be used to generate the Adomian polynomials for more complicated nonlinear terms, then the Adomian polynomials can be generated for all types of nonlinearities with facility on the computer, thus greatly enhancing our ability to solve nonlinear differential equations, encouraging physicists and engineers to consider much more realistic modeling from first physical principles without resort to dubious mathematical simplifications or cumbersome ad hoc methods merely for the sake of tractability.

CONCLUSIONS

In this paper we have reviewed the Adomian decomposition method for nonlinear ordinary differential equations with a general operator involving a special parameter λ. Further we have investigated the basis of Adomian's approach in the historical milieu of solvability of parameterized differential equations. We emphasize the Murray-Miller theorem of existence and uniqueness as integral to the underpinnings to Adomian's approach. We also studied the Taylor series expansion of the general nonlinear term as represented by an analytic function $f(u)$. We have shown that the Adomian decomposition series is theoretically derived from a parameterized Taylor expansion series only when the analytic parameter λ equals one. This study has explained Adomian's choices and defined the Adomian polynomials utilizing the Kronecker delta function in order to avail ourselves of the powerful symbolicprogramming software packages such as MATHEMATICA. This study has shown that Adomian's approach is a special choice corresponding to $\lambda=1$. This work also details a general procedure for generating the Adomian polynomials for a wide range of nonlinear functions limited to the requirements of analyticity as guaranteedby the local existence and uniqueness theorems by Picard and Lindelof, Cauchy and Kovalevskaya, and Murray and Miller. It is also worth mentioning that the recurrence relations may be found or given in different forms. In general their forms depend on the type of general operator and the structure of the problem under investigation. Finally we discuss the alternate choice of $\lambda(x) \neq 1$ as a fascinating new challenge for researchers, which introduces a general approach for the Adomian decomposition method.

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ON EINSTEIN-KAEHLERIAN CONHARMONIC* RECURRENT SPACES

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ABSTRACT

Mathai [2] and Walker [7] have studied Kaehlerian recurrent spaces and Ruse's spaces of recurrent curvature respectively, Singh and Singh [4] have defined and studied Kaehlerian conharmonic recurrent and Kaehlerian conharmonic symmetric spaces and several theorems have been established. Further, Singh [6] has defined and studied Einstein-Kaehlerian conharmonic recurrent spaces and several theorems have been investigated, Singh [11] has defined and studied Einstein-Kaehlerian projective recurrent spaces and several theorems have been obtained. Also, Singh and Kumar [13] have defined and studied on Einstein-Tachibana conharmonic recurrent spaces and several theorems have been established. Later, Chauhan, Chauhan and Singh [14] have defined and studied on Einstein-Kaehlerian space with recurrent Bochner curvature tensor and several theorems have been obtained.

Here we define and study Einstein-Kaehlerian conharmonic* recurrent spaces and several theorems have been derived. The necessary and sufficient conditions for an Einstein-Kaehlerian conharmonic* recurrent space to be Kaehlerian recurrent and Einstein-Kaehlerian conharmonic recurrent have been investigated.

Keywords: Einstein-Kaehlerian Conharmonic*, Conharmonic, Bochner, Projective, Recurrent, Space.

Classification Number : 53Bxx, 32C15, 46A13, 46M40, 53B35, 53C55.

INTRODUCTION

An $n\,(=2m)$ dimensional Kaehlerian space in a Riemannian space which admits a structure tensor field $F^h_{\ i}$ satisfying :

$$F^h_{\ j} F^i_{\ h} = -\delta^i_{\ j}, \tag{1.1}$$

$$F_{ij} = -F_{ji}, \ (F_{ij} = F^a_{\ i}) \tag{1.2}$$

and

$$F^h_{\ i,j} == 0 \tag{1.3}$$

where the comma (,) followed an index denotes the operator of covariant differentation with respect to the metric tensor g_{ij} of the Riemannian space.

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The Riemannian curvature tensor field is defined by

$$R^h{}_{ijk} = \partial_i\left\{{h \atop l\;k}\right\} - \partial_j\left\{{h \atop i\;k}\right\} - \left\{{h \atop i\;l}\right\}\left\{{l \atop j\;k}\right\} - \left\{{h \atop j\;l}\right\}\left\{{l \atop i\;k}\right\}, \tag{1.4}$$

whereas

$$R_{ij} = R^h{}_{hij} \text{ and } R = R_{ij}\, g^{ij}.$$

The Ricci tensor and the scalar curvature are respectively.

The Kaehlerian conharmonic curvature tensor $T^h{}_{ijk}$, is given by [3] :

$$T^h{}_{ijk} = \mathrm{R}^{\mathrm{h}}{}_{\mathrm{ijk}} + \frac{1}{n+4}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + g_{ik}R^h_j - g_{jk}R^h_i$$

$$+ + S_{ik}F^h_j - S_{jk}F^h_i + F_{ik}S^h_j - F_{jk}S^h_i + 2S_{ij}\;F^h_k + 2F_{ij}S^h_k) \tag{1.5}$$

And, the Kaehlerian chonarmonic* curvature tensor $*\,T^h{}_{ijk}$ is given by [8]

$$*T^h{}_{ijk} \underset{=\!=}{def} R^h_{ijk} + \frac{1}{n-2}(g_{ik}R^h_j - g_{jk}R^h_i) \tag{1.6}$$

where $S_{ij} = F^a{}_i R_{aj}$.

Let us suppose that a kaehlerian space is an Einstein one, then the Ricci tensor satisfies

$$R_{ij} == R/n\;\; g_{ij},\; R_{,a} == 0\,. \tag{1.7}$$

from which, we have

$$R_{ij.a} == 0,\; S_{ij.a} == 0, \text{ and } S_{ij} == R/n\; F_{ij}\,. \tag{1.8}$$

If a Kaehlerian space is an Einstein one, the Kaehlerian conharmonic curvature tensor and Kaehlerian conharmonic* curvature tensor reduce to.

$$E^h_{ijk} = R^h_{ijk} + \frac{2R}{n(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ik}F^h_k)\,. \tag{1.9}$$

and

$$**T^h_{ijk} = R^h_{ijk} + \frac{R}{n(n-2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) \tag{1.10}$$

respectively.

In view of equations (1.9) and (1.10), we obtain

$$E^h_{ijk} == {**}T^h_{ijk} + \frac{(n-8)R}{n(n-2)(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i)$$

$$+\frac{2R}{n(n+4)}(F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{1.11}$$

We shall use the following:

Definition (1.1) : A Kaehler space is said to be recurrent, if we have [1])

$$R^h{}_{ijk,a} - \lambda_a R^h{}_{ijk} = 0, \tag{1.12}$$

For some non -zero recurrence vector λ_a and is called Kaehlerian Ricci- recurrent, if it satisfies the relation

$$R_{ij,a} - \lambda_a R_{ij} = 0, \tag{1.13}$$

Multiplying the above equation by g^{ij}, we get

$$R,_a - \lambda_a R = 0. \tag{1.14}$$

Remark (1.1) : From (1.13), it follows that every Kaehlerian recurrent space is Kaehlerian Ricci -recurrent but the converse is not necessarily true.

Definition (1.2) : A Kaehler space satisfying the relation [6]

$$E^h_{ijk.a} - \lambda_a E^h_{ijk} = 0 \tag{1.15}$$

where λ_a is a non-zero recurence vector, will be called an Einstein-Kaehlerian conharmonic recurrent space or briefly an *E-K* * space.

Definition (1.3): A Kaehler space is called a space of constant holmorphic sectional curvature tensor. If the tensor $U^h{}_{ijk}$ defined by [5].

$$U^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{1.16}$$

vanishes identically.

EINSTEIN- KAEHLERIAN CONHARMONIC* RECURRENT SPACES

Defination (2.1) : A Kaehler space satisfying the relation:

$$**T^h_{ijk.a} - \lambda_a **T^h_{ijk} = 0 \tag{2.1}$$

where λ_a is a non -zero recurrence vector, will be called an Einstein- Kaehlerian conharmonic* recurrent space or briefly an *E-T** space.

We have the follwing:

Theorem (2.1): A necessary and sufficient condition for an *E-T** space to be a Kaehlerian recurrent is that the scalar curvature be equal to zero.

Proof: Suppose that an *E-T** Space is Kaehlerian recurrent, making use of equations (1.7), (1.8) and (1.10) in (2.1), we get

$$R^h_{ijk.a} = \lambda_a [R^h_{ijk} + \frac{R}{n(n-2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i] \tag{2.2}$$

Since, an *E-T** space is Kaehlerian recurrent, equation (2.2) reduces to

$$\frac{R}{n(n-2)}(g_{ik}\ \delta^h_j - g_{jk}\delta^h_i) = 0 \tag{2.3}$$

which gives $R = 0$

Conversely, if an *E-T** space satisfied $R = 0$, then equation (2.2) reduce to

$$R^h_{ijk.a} - \lambda_a R^h_{ijk} = 0$$

which shows that the space is Kaehlerian recurrent. This completes the proof.

Similary, is view of Theorem (2.1) and equations (1.7), (1.8), (1.9) and (1.15), we can prove the following:

Theorem (2.2): A necessary and sufficient condition for an *E-K** space to be Kaehlerian recurrent is that the scalar curvature be equal to zero.

Theorem (2.3): An *E-T** space is an *F-K** space iff the scalar curvature R is equal to zero.

Proof : Suppose that an *E-T** space is a an *E-K** space. Differentiating (1.11) covariantly with respect x^a and using (1.7) we obtain

$$E^h_{ijk.a} = **T^h_{ijk.a} \tag{2.4}$$

Multiplying (1.11) by λ_a and subtracting from (2.4), we have

$$E^h_{ijk.a} - \lambda_a E^h_{ijk} = **T^h_{ijka} - \lambda_a **T^h_{ijk} \frac{\lambda_a(n-8)R}{n(n-2)(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i\ )$$

$$-\frac{2\lambda_a R}{n(n+4)}(F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h) \qquad (2.5)$$

Now, making use of the above supposition, equation (2.5) reduces to

$$\lambda_a R[\frac{(n-8)}{n(n-2)(n+4)}(g_{ik}\ \delta_j^h - g_{jk}\delta_i^h) + \frac{2}{n(n+4)}\left(F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right)] = 0$$

which implies that $R = 0$.

Conversely, let us suppose that in an E-T^* space, the scalar curvature $R = 0$. Hance equation (2.5) reduces to

$$E^h_{ijk.a} - \lambda_a E^h_{ijk} = 0$$

which shows that the space is an E-K^* space. This completes the proof.

Theorem (2.4) : If a Kaehler space satisfies any two of the following:

(i) the space is an E-K^* space,

(ii) the space is an E-T^* space,

(iii) the scalar curvature is equal to zero,

It must also satisfy the third.

Proof: An E-K^* and an E-T^* spaces are characterized respectively by equations (1.15) and (2.1).

The statement of the theorem follows in view of equations (1.15), (2.1) and (2.5).

We have the following lemmas from Walker [7], and Yano and Bochner [10].

Lemma (2.1): The curvature tensor R_{hijk} satisfies the identity

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0. \qquad (2.6)$$

where $R^h{}_{hijk,l,m} \overset{def}{=} R^h{}_{hijk,lm}$

Lemma (2.2): If $a_{\alpha\beta}, b_\gamma$ are quantities satisfying

$$a_{\alpha\beta} == a_{\beta\alpha} \quad \text{and} \quad a_{\alpha\beta}b_\gamma + a_{\beta\gamma}b_\alpha + a_{\gamma\alpha}b_\beta = 0 \qquad (2.7)$$

For $\alpha, \beta, \gamma = 1, 2,$ ---------, N, then either all the $a_{\alpha\beta}$ are zero or all the b_γ are zero Making use of the above Lemmas, we shall prove the following :

Theorem (2.5): In an *E-T** space, either recurrence vector is gradient or the space is of constant holomorphic sectional curvature.

Proof: Differentiating (2.2) covariantly with respect to x^b and using equations (1.3), (1.7), (1.8) and (1.10) we get

$$R^{h}{}_{ijk,ab} = (\lambda_{a,b} + \lambda_a \lambda_b) **T^{h}_{ijk} \tag{2.8}$$

where $R^{h}{}_{ijk,a,b} \overset{def}{=} R^{h}{}_{ijk,ab}$

Multiplying (2.8) with $g_{hl,}$ we obtain

$$R_{ijkl,ab} == (\lambda_{a,b} + \lambda_a \lambda_b) **T_{ijkl}. \tag{2.9}$$

From (2.9) and the identity (2.6), we get

$$\lambda_{ah} **T_{ijkh} + \lambda_{ij} **T_{khab} + \lambda_{kh} **T_{ahij} = 0. \tag{2.10}$$

where $\lambda_{ab} \overset{def}{=} \lambda_{b,a} - \lambda_{a,b}.$

Equation (2.10) is of the form (2.7), since $**T_{ijkh} == **T_{khij}.$

Thus, from Lemma (2.2), we have Theorem (2.5).

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A COMMON FIXED POINT THEOREM FOR A SEQUENCE OF MAPPINGS IN MENGER SPACES

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ABSTRACT

In the present paper, we prove a unique common fixed point theorem for a sequence of mappings in Menger spaces. Our result improves and modifies a result of B.S. Choudhary, in the sense that, the commutativity of the mappings has not been considered and contraction condition taken in a different way.

Keywords: sequence of mappings, Menger spaces, common fixed point.

AMS Subject Classification (2000): 54H25, 47H10, 54E70.

INTRODUCTION AND PRELIMINARIES

Sehgal [5] initiated the study of contraction mappings on probabilistic metric spaces (PM-spaces) (cf. also [6] and [7]). Since then many fixed point results for single and multivalued mappings on PM-spaces are obtained. Recently Choudhary [1] has established a unique common fixed point theorem for a sequence of mutually contractive and commuting sequences of self-mappings on Menger spaces. In the present paper we modify and improve the result of Choudhary [1] for a sequence of self-mappings under a different contraction condition and not using the commutativity of mappings.

We now recall some basic definitions and results. A mapping F: R→ [0, 1] is called a distribution function if it is non-decreasing, left continuous $\sup_{x \in R} F(x) = 0$ and $\sup_{x \in R} F(x) = 1$

DEFINITION 1 [2]

A PM-space is an ordered pair $(X, \mathcal{F})$ where X is a nonempty set of elements and $\mathcal{F}$ is a mapping from $X \times X$ to £, the collection of all distribution functions. The value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The functions $F_{u,v}$ are assumed to satisfy the following conditions:

$(a)\ F_{u,v}(x) = 1$ for all $x > 0$ iff $u = v$;

$(b)\ F_{u,v}(0) = 0$;

$(c)\ F_{u,v} = F_{v,u}$;

$(d)\ F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$

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The following example of PM-space is cited from [3].

Example:

Let $X = \{p, q, r\}$. Define

$$F_{p,r}(x) = F_{r,p}(x) = F_{r,q}(x) = F_{q,r}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{2} & \text{if } 0 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

and

$$F_{p,q}(x) = F_{q,p}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{2} & \text{if } 0 < x \le \frac{3}{2} \\ 1 & \text{if } x > \frac{3}{2} \end{cases}$$

It is straightforward to show that $(X, \mathcal{F})$ is a PM-space.

DEFINITION 2 [4]

A mapping t: $[0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm (or T-norm) if it satisfies:

(e) $t(a, 1) = a$, $t(0, 0) = 0$;

(f) $t(c, d) \ge t(a, b)$ for $c \ge a$, $d \ge b$;

(g) $t(a, b) = t(b, a)$;

(h) $t(t(a, b), c) = t(a, t(b, c))$;

for all a, b, c, d in $[0, 1]$.

DEFINITION 3 [4]

A Menger space is a triplet (X, F, t), where (X, F) is a PM-space and t-norm t is such that the inequality

$$\text{(d')} \qquad F_{u,w}(x+y) \ge t\{F_{u,v}(x), F_{v,w}(y)\}$$

holds for all u, v, w X and all $x \ge 0$, $y \ge 0$.

Note that among a number of possible choices for t, $t(a, b) = \min\{a, b\}$ or simply "$t = \min$" is the strongest possible universal t (cf. [4, page 318]). Moreover, if t satisfies $t(x, x) \ge x$ for every $x \in [0, 1]$ then (f) and (d') imply

$$\text{(d'')} \qquad F_{u,w}(x+y) \ge \min\{F_{u,v}(x), F_{v,w}(y)\}$$

for all u, v, w X and all $x \geq 0, y \geq 0$.

Due to the simplicity and universality of "$t = \min$", (d") will be used frequently. For the details of the topological preliminaries refer to Schweizer and Sklar [4].

A sequence $\{u_n\}$ in X is said to converge to a point $z \in X$ if for given $\varepsilon > 0, \lambda > 0$, we can find a positive integer $N(\varepsilon, \lambda)$ such that

$$F_{u_n,z}(\varepsilon) > 1 - \lambda \text{ for all } n > (\varepsilon, \lambda). \text{ Or, equivalently } \lim_{n \in \infty} F_{u_n,z}(\varepsilon) = 1.$$

DEFINITION 4 [4]

A sequence $\{u_n\}$ in a PM-space is a Cauchy sequence if for every $\varepsilon > 0, \lambda > 0$, there exists an integer $N(\varepsilon, \lambda)$ such that $F_{u_n}, {}_{u_m}(\varepsilon) > 1 - \lambda$ for all $n, m \geq N(\varepsilon, \lambda)$. A PM-space is complete if every Cauchy sequence in it converges to some point of the space. The definition of a Cauchy sequence is extended and modified via the following Lemma which is due to Singh and Pant [8].

LEMMA 1

Let $\{u_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$, where t is continuous and $t(x, x) \geq x$ for every $x \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that $F_{u_n,u_{n+1}}(kx) \Rightarrow F_{u_{n-1},u_n}(x)$ for all n, then $\{u_n\}$ is a Cauchy sequence in X.

MAIN RESULT

Choudhary [1] prove the following result.

THEOREM C

Let $(X, \mathcal{F}, t)$ be a complete Menger space and $\{T_i\}$ $i \in N$ be a sequence of self-mappings on X such that

(a) T_i is continuous for all $i \in N$.

(b) for $x > 0, 0 < k < 1$,

$$F_{T_i u T_j v}(x) \geq F_{u,v}(x / k)$$

where $u, v \in X, i \neq j$ and $u \neq v$.

(c) $T_i T_j = T_j T_i$ for all $i, j \in N$.

Then $\{T_i\}_{i \in N.}$ has a unique common fixed point.

In the present paper, we prove the following result without using the condition (c) and taking condition (b) in a different way.

THEOREM 1

Let (X, F, t) be a complete Menger space, where t is continuous satisfies $t(x, x) \geq x$ for every $x \in (0,1)$. Let $\{T_i\}_{i \in N}$ be a sequence of self-mappings on X such that

each T_i is continuous, $i \in N$ (1)

for each $u, v \in X$, $i, j \in N$, $u \neq v$, $i \neq j$, $x > 0$, $0 < k < 1$, $1 < \alpha \leq 2$. (2)

$$F_{T_i u, T_j v}(x) \geq \min\{F_{T_i u, u}(x/k), F_{T_j v, v}(\alpha x/k) F_{u,v}(x/k)\}$$

and $F_{T_i u, T_i v}(x) \geq F_{u,v}(x/k)$. (3)

Then the family $\{T_i\}$ have a unique common fixed point.

PROOF

For any $u_0 \in X$, we can choose a point $u_1 \in X$ such that $u_1 = T_1 u_0$. Similarly, we can choose an $u_2 \in X$ such that $u_2 = T_2 u_1$. In general, we can construct a sequence $\{u_n\}$ in X such that $u_n = T_n u_{n-1}$.

Now, the following cases may arise:

Case 1: Let us assume that $u_n \neq u_{n-1}$, then for any $x > 0$ from (2),

$$F_{u_n, u_{n+1}}(x) = F_{T_n u_{n-1}, T_{n+1} u_n}(x)$$

$$\geq \min\{F_{T_n u_{n-1}, u_{n-1}}(x/k), F_{T_{n+1} u_n u_n}(\alpha x/k), F_{u_{n-1} u_n}(x/k)\}$$

$$= \min\{F_{u_n, u_{n-1}}(x/k) F_{u_{n+1} u_n}(\alpha x/k)\}$$

$$= F_{u_{n-1}, u_n}(x/k) \text{ since } F_{u_n, u_{n+1}}(x) \leq F_{u_n, u_{n+1},}(x)(\alpha x/k),\ k < 1$$

Therefore, by Lemma 1, $\{u_n\}$ is a Cauchy sequence in X. Since X is complete, $\{u_n\}$ converges to some point z in X, Thus for $x > 0, \lambda > 0$, there exists an integer $N(x,\lambda)$ such that

$F_{u_n,z}(x/2) > 1-\lambda$ for all $n > N$. Or, equivalently, $\lim\limits_{n\to\infty} F_{u_n,z}(x) = 1$ (4)

Now for $n > i$, taking $\alpha = 2$, from (d”) and (2),

$$F_{T_i z,z}(x) \ge \min\{F_{u_n,z}(x/2), F_{T_i z,u_n}(x/2)\}$$

$$= \min\{F_{u_n,z}(x/2), F_{T_i z, T_n u_{n-1}}(x/2)\}$$

$$\ge \min\{F_{u_n,z}(x/2), \min\{F_{T_n u_{n-1} u_{n-1}}(x/2k), F_{T_i z,z}(x/k), F_{u_{n-1},z}(x/2k)\}\}$$

Taking the limit $n \to \infty$ and using (4),

$$F_{T_i z,z}(x) \ge \min\{1-\lambda, 1, F_{T_i z,z}(x/k) = 1-\lambda$$

Hence $T_i z = z$ for all $i \in N$

Case II: Now suppose that $u_i = u_{i-1}$ for some integer i, then

$$u_{i-1} = T_i u_{i-1}$$

Let $z = u_i$ which implies that $T_i z = z$ Let further $z \ne T_j z$ for some j.

Assume that, $z \ne T_j^n z$ for all $n \in N$, then for $x > 0$, taking $\alpha = 1+k$ and using (2)

$$F_{z,T_j^2 z}(x) = F_{T_{iz},T_{j(T_{jz})}}(x)$$

$$\ge \min\{F_{T_i z,z}(x/k), F_{T_j^2 z,T_j z}((1+k)x/k), F_{z.T_j z}(x/k)\}$$

$$\ge \min\{1, F_{T_j^2 z,z}(x), F_{z,T_j z}(x/k)\}$$

$$= F_{z,T_j}(x/k)$$

Using above argument repeatedly, we have

$$F_{z,T_j^n z}(x) \geq F_{z,T_j z,}(x/k^{n-1}) \quad n = 2,3 \ldots\ldots\ldots\ldots\ldots\ldots$$

Further, we observe that

$$T_i^n z \to z \text{ as } n \to \infty .$$

since T_j is continuous, so $T_j^{n+1} z \to T_j z$ as $n \to \infty$

Therefore $z = T_j z$ for all $j \in N$, which is a contradiction. Thus $z = T_j^p z$ for some p.

Let p be the smallest integer such that $z \neq T_j^m z$ for $m = 1, 2 . . p-1$ (5)

Then from (2) and (3), taking $\alpha = 1 + k$, we have

$$F_{z,T_j z}(x) = F_{T_j^p z, T_j T_i z}(x) = F_{T_j(T_i z), T_j(T_j^{p-1} z)}(x)$$

$$\geq F_{T_i z, T_j^{p-1} z}(x/k) = F_{T_i z, T_j(T_j^{p-2} z)}(x/k)$$

$$\geq \min\{F_{z.T_i z}(x/k^2) \cdot F_{z.T_i z}(x/k^2) \cdot F_{T_j^{p-1} z T^{p-2},}((1+k)x/k^2), F_{z.T_j^{p-2} z}(x/k^2)\}$$

$$= F_{z,T_j^{p-2} z}(x/k^2) \geq \ldots\ldots\ldots \geq F_{z,T_i z}(x/k^{p-1})$$

is a contradiction

Thus $z = T_j z$ for all $j \in N$.

Uniqueness of the fixed point z easily follows from (2).

COROLLARY 1

Let (X,d) be a complete metric space and $\{T_i\}_{i \in N}$ be a sequence of self mappings on X such that

(a) T_i is continuous for all $i \in N$.

(b) for each $u, v \in X_j$, $i, j \in N, u \neq v, i \neq j$,

$$d(T_i u, T_j v) \leq k \max\{d(T_i u, u), \frac{1}{\alpha} d(T_j v, v), d(u, v)\}, 0 < k < 1, \text{ and}$$

(c) $d(T_i u, T_i v) \leq kd(u, v)$.

Then the family $\{T_i\}$ have a unique common fixed point.

The proof of the corollary is based on the fact that, a complete metric space may be regarded as a complete Menger space if we take $F_{u,v}(x) = H(x\text{-}d)(u,v)$ where

$$H(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{otherwise} \end{cases}$$

and $t(a,b) = \max\{a,b\}$, It may be seen that conditions (b) and (c) then imply conditions (2) and (3) respectively.

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DAMPED VIBRATIONS OF NON HOMOGENOUS CIRCULAR PLATE OF PARABOLICALLY VARYING THICKNESS RESTING ON ELASTIC FOUNDATION

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ABSTRACT

The research in the field of vibration is unceasingly acquiring immense importance in modern science as it has great significance in every field of applied science today. The effect of vibration on design of machine, multistoried building, radio-telescope, nuclear reactor technology, earthquake resistant structure and various other engineering structures is of immense importance. Today we are going ahead in space technology where various problems of plates continuously supported by elastic and visco-elastic media are considered. Also effect of damping is applied for practical use.

In this paper Damped vibrations of Non homogenous circular plate of parabolically varying thickness resting on elastic foundation have been studied. The fourth order differential equation of motion is solved by the method of Frobenius. Here the transverse displacement equation of motion is solved by the method of Frobenius. Here the transverse displacement is expressed as an infinite series in terms of radial coordinates. The frequencies, deflection functions and moment parameters corresponding to the first two modes of vibrations are computed for the circular plate with clamped & simply supported edge conditions for various values of taper constants and damping parameters viz. Non homogeneity, Damping, elastic foundation.

Keywords: Non homogeneity, damping, elastic foundation, taper constant

INTRODUCTION

A mathematical model of Damped vibrations of Non homogenous circular plate of parabolically varying thickness resting on elastic foundation have been studied on the basis of classical plate theory. The fourth order differential equation of motion is solved by the method of Frobenius.

Here, again the transverse displacement is expressed as an infinite series in terms of radial coordinates. The frequencies, deflection functions and moment parameters corresponding

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to the first two modes of vibrations are computed for the circular plate with clamped & simply supported edge conditions for various values of taper constants and damping parameters viz. Non homogeneity, Damping, elastic foundation.

EQUATION OF MOTION

The governing differential equation of the transverse motion of a Non homogenous plate of variable thickness under damping and elastic foundation given as

$$\nabla^2 (D\nabla^2 w) - (1-\nu)\left(\frac{\partial^2 D\partial^2 w}{\partial y^2\,\partial x^2} - \frac{\partial^2 D\partial^2 w}{\partial x\partial y\,\partial x\partial y} + \frac{\partial^2 D\partial^2 w}{\partial x^2\partial y^2}\right) + \rho h\frac{\partial^2 w}{\partial t^2} + \frac{K\partial w}{\partial t} + K_F w = 0 \quad (1)$$

where ∇^2 is the two dimensional Laplacian operator in Cartesian coordinates w is transverse deflection, ν is the Poisson ratio and D is the flexural rigidity at any point of the plate, ρ is the mass density per unit volume h is thickness of plate, t is the time, K is damping constant and K_F is the foundation modulus. After changing the above equation to polar coordinates and introducing the non dimensional variables, $H = \frac{h}{a}, X = \frac{r}{a}, W = \frac{w}{a}$, and if thickness of plate H, modulus of elasticity and mass density are assumed to vary in the form,

$$H = H_0(1-\alpha X^2)$$
$$\overline{E} = E_0(1+\beta X)$$
$$\overline{\rho} = \rho(1+\beta X)$$

where α is taper constant, β is non homogeneity parameter one obtains,

$$(1-\beta X)(1-\alpha X^2)^3\frac{\partial^4 W}{\partial X^4} + 2\left[(1-\alpha X^2)^3\beta + \left\{-6\alpha X(1-\alpha X^2)^2 + \frac{(1-\alpha X^2)^3}{X}\right\}(1+\beta X)\right]\frac{\partial^3 W}{\partial X^3}$$

$$+\left[\left\{-12\alpha X(1-\alpha X^2)^2 + \frac{(2+\nu)(1-\alpha X^2)^3}{X}\right\}\beta\right]\frac{\partial^2 W}{\partial X^2} +$$

$$\left[\left\{-6\alpha(1-\alpha X^2)^2 + 24\alpha^2 X^2(1-\alpha X^2) - \frac{(1-\alpha X^2)^3}{X^2} - 6\alpha X\frac{(2+\nu)(1-\alpha X^2)^2}{X}\right\}(1+\beta X)\right]\frac{\partial^2 W}{\partial X^2}$$

$$+\left[\left\{-12\alpha\, X\nu\frac{(1-\alpha X^2)^2}{X}-\frac{(1-\alpha X^2)^3}{X}\right\}\beta\right]\frac{\partial W}{\partial X}+$$

$$\left[\left\{-6\alpha\upsilon\,\frac{(1-\alpha X^2)^2}{X}-+24\alpha^2 X\upsilon(1-\alpha X^2)^2+6\alpha\frac{(1-\alpha X^2)^2}{X}+\frac{(1-\alpha X^2)^3}{X^3}\right\}(1-\beta X)\right]\frac{\partial W}{\partial X}+$$

$$\frac{(1-\alpha X^2)^3\,(1-\beta X)\,(1-\upsilon^2)a^2\rho_0}{\overline{E_0}\;\; I^*}\frac{\partial^2 W}{\partial t^2}+\frac{F_D}{C^*}\frac{\partial W}{\partial t}+\frac{F_p}{C^*}W=0 \qquad (2)$$

where

$$F_p=\frac{K_F}{E_0},\;\; F_d=\frac{K_D a}{E_0},\, C^*=\frac{H_0^3}{12(1-\mathrm{v}^2)},\, I^*=\frac{H_0^2}{12}$$

SOLUTION FOR DAMPED HAMONIC VIBRATIONS

For Damped Harmonic Vibrations, the solution is given by

$$W=(X,t)=\overline{W}(X)e^{-rt}\cos pt$$

Substituting (3) in (2) & solving we get,

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$$(1-\beta X)^2(1-\alpha X^2)^4\frac{\partial^4\overline{W}}{\partial X^4}+2\left[(1-\alpha X^2)^4\beta(1+\beta X)+\left\{-6\alpha X(1-\alpha X^2)^3+\frac{(1-\alpha X^2)^4}{X}\right\}(1+\beta X)^2\right]\frac{\partial^3\overline{W}}{\partial X^3}$$

$$+\left[\left\{-12\alpha X(1-\alpha X^2)^3(1-\beta X)+\frac{(2+\upsilon)(1-\alpha X^2)^4(1-\beta X)}{X}\right\}\beta\right]\frac{\partial^2\overline{W}}{\partial X^3}$$

$$+\left[\left\{-6\alpha(1-\alpha X^2)^3+24\alpha^2X^2(1-\alpha X^2)^2-\frac{(1-\alpha X^2)^4}{X^2}-6\alpha\,(2+\upsilon)(1-\alpha X^2)^3\right\}(1+\beta X)^2\right]\frac{\partial^2\overline{W}}{\partial X^2}$$

$$\left[\left\{-12\alpha\upsilon(1-\alpha X^2)^3-\frac{(1-\alpha X^2)^4}{X}\right\}\beta(1+\beta X)\right]\frac{\partial\overline{W}}{\partial X}$$

$$+\left[\left\{-6\alpha\upsilon\frac{(1-\alpha X^2)^3}{X}+24\alpha^2 X\upsilon(1-\alpha X^2)^2+6\alpha\frac{(1-\alpha X^2)^3}{X}+\frac{(1-\alpha X^2)^4}{X^3}\right\}(1+\beta X)^2\right]\frac{\partial\overline{W}}{\partial X}$$

$$-\left[(1-\alpha X^2)^2(1+\beta X)^2 M^{*}\Omega^2+D_K M^{*2}-(1+\beta X)(1-\alpha X^2)E_F C^{*}\right]\overline{W}=0$$

where,

$$D_K=\frac{3(1-\upsilon^2)K^2}{\rho_0 E_0},\ M^{*}=\frac{1}{H_0^2},\ C^{*}=\frac{1}{H_0^3},\ E_F=\frac{12K_F(1-\upsilon^2)}{E_0}$$

$$\Omega^2=\frac{12(1-\upsilon^2)\rho_0 a^2 p^2}{E_0}$$

Here p is circular frequency, D_k is damping parameter, Ω is frequency parameter & E_F is a foundation Modulus parameter.

A Series solution $\overline{W}$ is assumed in the form $\overline{W}(X)=\sum_{\lambda=0}^{\infty}a_\lambda X^{C+\lambda},\ a_0\neq 0$, where c is exponent of singularity. Applying method of Frobenius we get the values of $c=0, 0, 2,2$ and we obtain solution for $\overline{W}$ corresponding to $c=0$, as,

$$\overline{W}=a_0\left[1+\sum_{\lambda=3}^{\infty}A_\lambda X^\lambda\right]+a_2\left[X^2\sum_{\lambda=3}^{\infty}+B_\lambda X^\lambda\right] \tag{5}$$

In the above solution, we see that no new solution will arise corresponding to the other values of c, as they are already contained in the solution given by equation (5). Also here values of A_λ and B_λ for are functions of α, β, D_K, E_F, M^*, N^* and Ω.

BOUNDARY CONDITIONS & FREQUENCY EQUATIONS

The frequency equations for clamped amd simply suppiorted circular plates have been obtained by employing the appropriate boundary conditions.

Clamped plate:

For a circular plate clamped at edges $r = a$, the deflection W and slope of the plate element at edges should be zero.

$$w(r,t)\big|_{r=a} = \frac{\partial w(r,t)}{\partial t}\big|_{r=a} = 0$$

Or

$$\overline{W}\big|_{X=a} = \frac{\partial \overline{W}}{\partial X}\big|_{X=1} = 0 \tag{6}$$

Using equation (5) and applying the boundary condition, (6) one obtains

$$a_0\left[1+\sum_{\lambda=3}^{\infty} A_\lambda\right] + a_2\left[1+\sum_{\lambda=3}^{\infty} B_\lambda\right] = 0$$

$$a_0\left[\sum_{\lambda=3}^{\infty} \lambda A_\lambda\right] + a_2\left[2+\sum_{\lambda=3}^{\infty} \lambda B_\lambda\right] = 0$$

Elimination the unknown constants a_0 and a_2, one obtains the frequency equation for (C-plate) as

$$\begin{vmatrix} V_1(\Omega) & V_2(\Omega) \\ V_3(\Omega) & V_4(\Omega) \end{vmatrix} = 0 \tag{7}$$

where as,

$$V_1(\Omega) = 1 + \sum_{\lambda=3}^{\infty} A_\lambda,$$

$$V_2(\Omega) = 1 + \sum_{\lambda=3}^{\infty} B_\lambda,$$

$$V_3(\Omega) = \sum_{\lambda=3}^{\infty} \lambda A_\lambda,$$

$$V_4(\Omega) = 2 + \sum_{\lambda=3}^{\infty} \lambda B_\lambda,$$

SIMPLY SUPPORTED PLATE

For a circular plate simply supported at the edge $r = a$, the deflection w and the moments M_r at the edge should be zero.

$$w\,(r,t)\big|_{r=a} = M_r\,(r,t)\big|_{r=a} = 0$$

Or,

$$\overline{W} = \left[\frac{\partial^2 \overline{W}}{\partial X^2} + \frac{\upsilon}{X}\frac{\partial \overline{W}}{\partial X}\right]_{X=1} = 0 \tag{8}$$

Applying these boundary conditions on the equation (5),one obtains

$$a_0\left[1 + \sum_{\lambda=3}^{\infty} A_\lambda\right] + a_2\left[1 + \sum_{\lambda=3}^{\infty} B_\lambda\right] = 0$$

$$a_0\left[\sum_{\lambda=3}^{\infty} \lambda(\lambda+\upsilon-1)A_\lambda\right] + a_2\left[2(1+\upsilon) + \sum_{\lambda=3}^{\infty} \lambda(\lambda+\upsilon-1)B_\lambda\right] = 0$$

Elimination the unknown constants a_0 & a_2 one gets the frequency equation for simply supported plate as,

$$\begin{vmatrix} V_1(\Omega) & V_2(\Omega) \\ V_5(\Omega) & V_6(\Omega) \end{vmatrix} = 0 \tag{9}$$

where

$$V_5(\Omega) = \sum_{\lambda=3}^{\infty} \lambda(\lambda+\upsilon-1)A_\lambda,$$

$$V_6(\Omega) = 2(1+\upsilon) + \sum_{\lambda=3}^{\infty} \lambda\,(\lambda+\upsilon-1)B_\lambda,$$

DEFLECTION FUNCTIONS & MOMENTS

The Transverse displacement function $\overline{W}$ (5) is given by:

$$\overline{W} = a_0\left[1 + \sum_{\lambda=3}^{\infty} A_\lambda X^\lambda\right] + a_2\left[X^2 + \sum_{\lambda=3}^{\infty} B_\lambda X^\lambda\right] \tag{10}$$

The unknown constants a_0 and a_2 are determined by applying boundary condition at $X = 1$. The equation thus obtained are such that only the ratio of the constants can be determined. We take $a_0 = 1$, and get the value of a_2 as,

$$a_2 = -\frac{1+\sum_{\lambda=3}^{\infty} A_\lambda}{1+\sum_{\lambda=3}^{\infty} B_\lambda}.$$

Substituting a_0 and a_2 the above equation (10) becomes.

$$\overline{W} = \left[1+\sum_{\lambda=3}^{\infty} A_\lambda X^\lambda\right] - \frac{1+\sum_{\lambda=3}^{\infty} A_\lambda}{\sum_{\lambda=3}^{\infty} B_\lambda}\left[X^2+\sum_{\lambda=3}^{\infty} B_\lambda X^\lambda\right] \tag{11}$$

Also,

$$M_r = -D\left[\frac{\partial^2 \overline{W}}{\partial r^2} + \frac{\upsilon}{r}\frac{\partial \overline{W}}{\partial r}\right]$$

Again enforcing the boundary condition $W = 0$ at $X = 1$ and sdopting the same value of a_0 and a_2 the non dimensional parameter is obtained in the from

$$\overline{M} = \frac{M_X}{D_0} = -(1-\alpha X^2)^3\left[\sum_{\lambda=3}^{\infty} \lambda\,(\lambda+\upsilon-1)A_\lambda X^{\lambda-2}\right] \tag{12}$$

$$+(1-\alpha X^2)^3 \frac{\sum_{\lambda=3} A_\lambda}{\sum_{\lambda=3}^{\infty} B_\lambda}\left[2(1+\upsilon)+\sum_{\lambda=3}^{\infty} \lambda(\lambda+\upsilon-1)B_\lambda X^{\lambda-2}\right]$$

where

$$D_0 = \frac{EH_0^3 a^3}{12(1-\upsilon^2)} \tag{13}$$

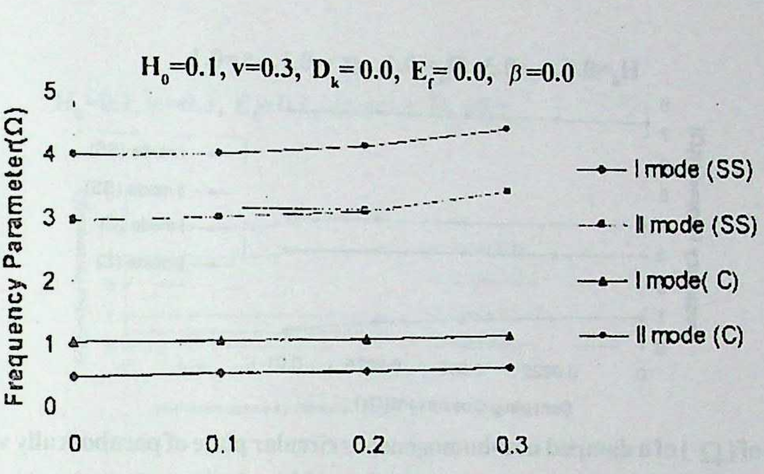
The values of Ω for both edge conditions have been taken from equation (7) and (9).

RESULTS & DISCUSSION

All the numerical results for a Damped isotropic non homogenous circular plate of linearly varying thickness considering foundation effect corresponding to first two modes of vibration for both clamped & simply supported edge condition have been computed using latest computer techniques. The result have been tabulated and graphically shown. Poisson ratio has been taken to be 0.3 and thickness of plate 0.1. Terms of the series up to an accuracy of 10^{-8} in their absolute values have been retained. By allowing damping parameter, non homogeneity parameter and elastic foundation to be zero, the problem of homogenous circular plate of parabolically varying thickness the result so obtained compare well with already published work of Gupta [4].

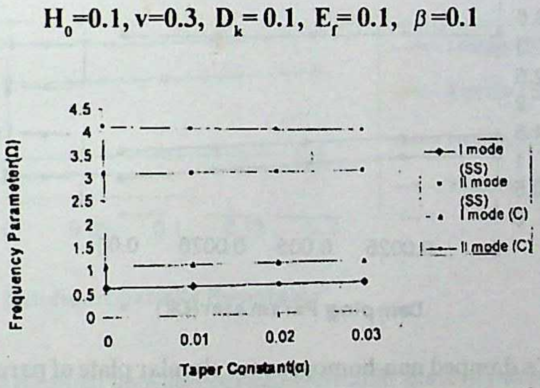
Figures (1) and (2) shows the variation of frequency parameter (Ω) with taper constant (α) for different values of Non homogeneity parameter, damping parameter & elastic foundation parameter for both clamped plate as well as simply supported plate. We see that on increasing the value of taper constant there is gradual decrease in frequency parameter in both in both of modes. Figures (3) and (4) shows the variation of frequency parameter (α) with foundation parameter (E_f) for different values of taper constant, non-homogeneity parameter & damping. There is increase in frequency parameter as we increase the foundation effect. This effect is greater for first mode than the second mode in both clamped plate & simply supported plate.

Figures (5) and (6) shows the variation of frequency parameter (Ω) with damping parameter (D_k) for different values of non homogeneity parameter, taper constant and Elastic foundation parameter. We observe that as we increase the damping effect the frequency parameter decreases. The effect of damping on first mode is more significant than for second mode in both the case i.e. clamped plate as well as for simply supported plate. Figures (7) and (8) shows variation of frequency parameter (Ω) with non homogeneity parameter (β) considering fixed value of taper constant Damping parameter, and foundation parameter. Deflection parameter & moment parameter have also been computed for both edge conditions corresponding to the first two modes of vibration. Figures (9) and (10) show the variation of deflection function & Moment parameter with respect to the different points on the plate surface from axis of symmetry.



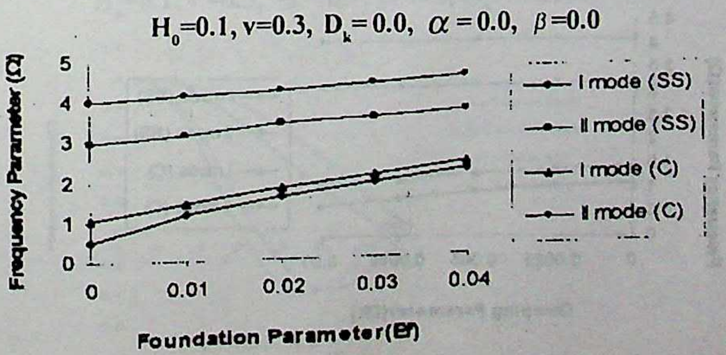
H_0=0.1, v=0.3, D_k= 0.0, E_f= 0.0, β=0.0

Fig.1: **Variation of (Ω) a damped non-homogeneous circular plate of parabolically varying thickness resting on elastic foundation for different values of taper constant.**



H_0=0.1, v=0.3, D_k= 0.1, E_f= 0.1, β=0.1

Fig- 2: **Variation of (Ω) a damped non-homogeneous circular plate of parabolically varying thickness resting on elastic on foundation for different values of taper constant.**



H_0=0.1, v=0.3, D_k= 0.0, α = 0.0, β=0.0

Fig- 3: **Variation of (Ω) a damped non-homogeneous circular plate of parabolically varying thickness resting on elastic foundation for different values of foundation parameter.**

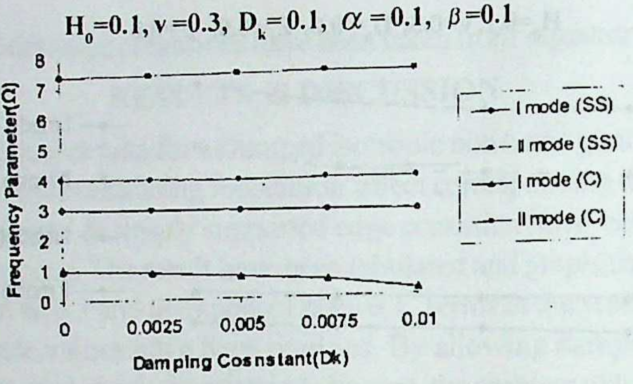


Fig- 4: Variation of (Ω) of a damped non-homogeneous circular plate of parabolically varying thickness resting on elastic foundation for different values of foundation parameter.

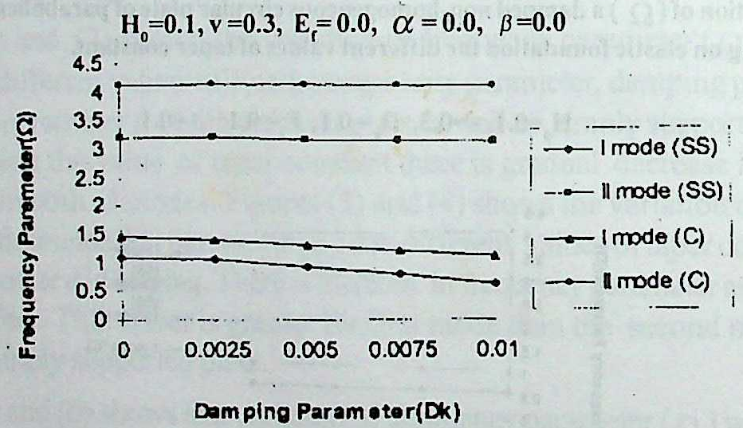


Fig- 5: Variation of (Ω) of a damped non-homogeneous circular plate of parabolically varying thickness resting on elastic foundation for different values of damping parameter.

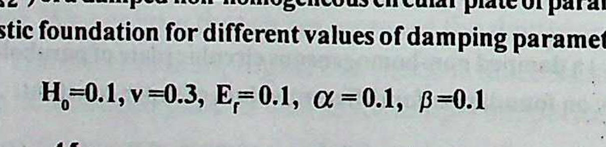


Fig- 6: Variation of (Ω) of a damped non-homogeneous circular plate of parabolically varying thickness resting on elastic foundation of different values of damping parameter.

H_0=0.1, v =0.3, E_f= 0.1, α = 0.1, D_k=0.1

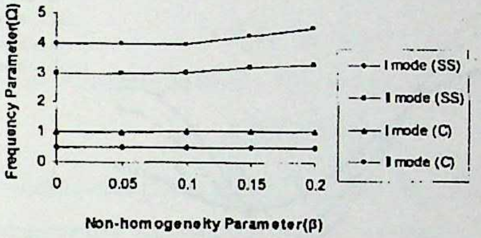


Fig- 7: **Variation of (Ω) of a damped non-homogeneous circular plate of parabolically varying thickness resting on elastic foundation of different values of non-homogeneity parameter.**

H_0=0.1, v =0.3, E_f= 0.1, α = 0.1, D_k=0.1

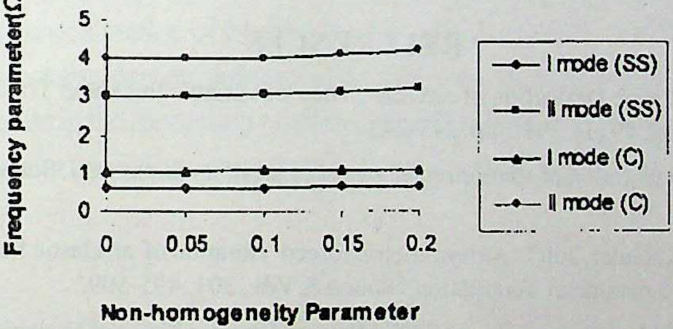


Fig- 8: **Variation of (Ω) of a damped non-homogeneous circular plate of parabolically varying thickness resting on elastic foundation of different values of non-homogeneity parameter.**

H_0=0.1, v =0.3, α = 0.1, β=0.1, D_k =0.01, E_f= 0.01

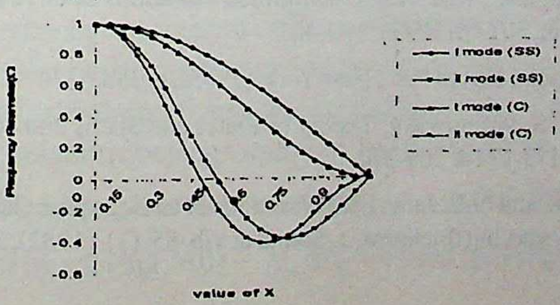


Fig- 9: **Transverse Deflection at different point of a Non-homogeneous damped Circular Plate of Parabolically varying thickness resting on elastic foundation.**

$H_0=0.1, \nu=0.3, \alpha=0.1, \beta=0.1, D_k=0.01, E_f=0.01$

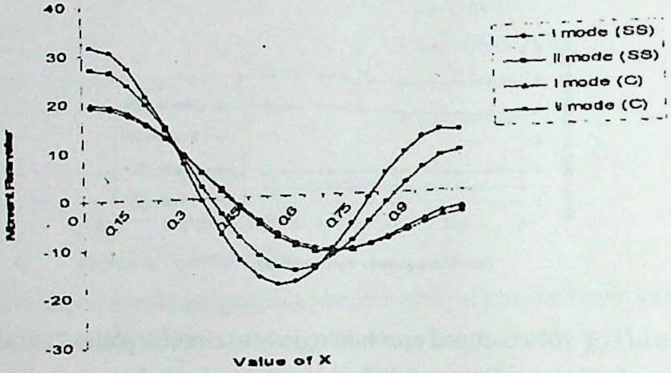


Fig- 10: Moment parameter (M) at different point of a No-homogeneous damped Circular Plate of Parabplically varying thickness resting on elastic foundation.

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STOCHASTIC ANALYSIS OF GOMPERTZ GOWRTH WITH PERIODICALLY VARYING GROWTH AND DEVELOPMENT PARAMETERS

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ABSTRACT

We present a stochastic analysis of a population obeying Gompertz growth equation with periodic growth coeffcient and development parameter under the influence of randomly fluctuating environment.

Keywords: Gompertz Equation, Periodic Growth Coefficien, Periodic Development Parameter, Fluctuation.

INTRODUCTION

The growth is an important significant part of life processes [2,9]. Gompertz [4] model equation which is generalization of Malthusian [6] growth model has found wide application in different growth processes of men, animals, plants, tumour etc. [5]. Gompertz quardratic equation plays a significant role in the modelling of plants growth. It is of vital importance in agriculture and forestry [8]. Gompertz model equation for single-species population is given by [1]

$$\frac{dN(t)}{dt} = a_0\, e^{-kt} N(t) \tag{1.1}$$

where $N(t)$ is the population size at any time t, a_0 is the growth coefficient, k is the development or senescence parameter. In the above model equation both the parameters a_0 and k are constants. A great phenomena are cyclic or oscillatory in nature. Due to diurnal or annual changes, there are numerous environmental factors like themperature, sunlight, soil or moisture, nutrients etc. cause annual cyclic variations of the growth and development parameters of trees [1,15]. Besides, the randomly fluctuating environment also influences the growth process [10-14,16,17]. In recent years interest on the mathematical modelling of biological and tumor growths based on Gompertz equation and its present paper is to study the overall effects of the periodically varying growth coefficient and the randomly fluctuating environment of the growth behaviours of a population obeying Gompertz model equation. The equation (1.1) can not be analysed directly by adding fluctuating term in a_0, k and in the whole system. So we slightly change this model by solving $N(0) = N_0$ at $t = 0$ then by simplification we get the changed model as follows :

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$$\frac{d\ln(N(t)/N_0)}{dt} = -k\ln(N(t)/N_0) + a_0$$

or

$$\frac{dy(t)}{dt} = -k\,y(t) + a_0 \tag{1.2}$$

where

$$y(t) = \ln N(t) - \ln N_0 \tag{1.3}$$

$y(t)$ is increasing function as $N(t)$ is increasing function.

GOMPERTZ GROWTH WITH SINUSOIDALLY VARYING GROWTH COEFFICIENT IN RANDOM ENVIRONMENT

Now we shall start with a simple harmonic growth coefficient

$$a_0(t) = \alpha_0 + \alpha\sin\omega t; \quad \alpha_0,\ \alpha > 0 \tag{2.1}$$

This implies that the time dependent growth coefficient $a_0(t)$ has average value a_0 and oscillation with amplitude α about this average value at frequency $\omega = 2\pi/T$, T is theperiod of oscillation. The oscillation increases as α increases and no oscillation for $\alpha = 0$. For small oscillation in $a_0(t), \alpha_0$ and α satisfy the condition:

$$\left|\frac{\alpha}{\alpha_0}\right| << 1 \tag{2.2}$$

We have considered here k is constant. To take account of the influence of the randomly fluctuating environment on the growth parameter $a_0(t)$, we modify equation (2.1) by adding a random perturabation or force $\varepsilon(t)$ satisfying the conditions [3].

$$<\varepsilon(t)>0, <\varepsilon(t)\varepsilon(t')> = \delta(t-t') \tag{2.3}$$

The sign <> denotes the average over the ensemble of stochastic forces. Then (2.1) becomes

$$a_0(t) = \alpha_0 + \alpha\sin\omega t + \varepsilon(t). \tag{2.4}$$

Putting equation (2.4) in (1.2) we get

$$\frac{dy(t)}{dt} = -k\,y(t) + \alpha_0 + \alpha\sin\omega t + \varepsilon(t) \tag{2.5}$$

which is a stochastic differential equation. According to Gardiner [3] equation (2.5) is the form of

$$\frac{dy(t)}{dt} = \{a(t) + b(t)\,y(t)\} + \{h(t) + g(t)\,y(t)\}\varepsilon(t)$$

where $a(t) = \alpha_0 + \alpha \sin \omega t$, $b(t) = -k$, $h(t) = 1$, $g(t) = 0$ and properties of $\varepsilon(t)$ are given by equation (2, 3), we use for n th order moment equation

$$\begin{aligned}\frac{d < y^n(t) >}{dt} = {} & < y^n(t) > \left[n\,b(t) + \frac{n(n-1)}{2} g^2(t)\right] \\ & + < y^{n-1}(t) > \left[n\,a(t) + n(n-1)h(t(g(t))\right] \\ & + < y^{n-2}(t) > \left[\frac{n(n-1)}{2} h^2(t)\right]\end{aligned} \tag{2.6}$$

Taking $n = 1$ in equation (2.6) the mean $< y(t) >_1$ is then given by

$$\frac{d < y(t) >_1}{dt} + k < y(t) >_1 = \alpha_0 + \alpha \sin \omega t \tag{2.7}$$

The solution of the equation (2.7) is given by

$$\begin{aligned}< y(t) >_1 = {} & \left(< y(0) >_1 - \frac{\alpha_0}{k} + \frac{\alpha\omega}{k^2+\omega^2}\right)e^{-kt} + \frac{\alpha_0}{k} \\ & \frac{+\alpha(k \sin \omega t - \omega \cos \omega t)}{k^2+\omega^2}\end{aligned} \tag{2.8}$$

where $< y(0) >_1$ is the initial value of $< y(t) >_1$. So that $< y(t) >_1$ is exponentially and periodically increases with time t but for large t, $< y(t) >_1$ tends to be a finite value with oscillation. For the graph of the equation (2.8) we shall take $\alpha_0 = 0.280$, $\alpha = 0.520$, $k = 0.0747$, $\omega = 0.524$, $T = 12$ months, $< y(0) >_1 = 1.5$. The graph of equation (2.8) for average $< y(t) >_1$ has been shown time (t) versus $< y(t) >_1$.

The equation of second order moment $< y^2(t) >_1$ is given by putting $n = 2$ in equation (2.6), then after calculation

$$\frac{d<y^2(t)>_1}{dt}+2k<y^2(t)>_1=2(\alpha_0+\alpha\sin\omega t)$$

$$\times\left\{Ae^{-kt}+\frac{\alpha_0}{k}+\frac{\alpha(k\sin\omega t-\omega\cos\omega t)}{k^2+\omega^2}\right\} \tag{2.9}$$

where

$$A=<y(0)>_1-\frac{\alpha_0}{k}+\frac{\alpha\omega}{k^2+\omega^2}$$

with intial condition $<y(0)>_1=\sigma_1^2(0)>_1$ at t = 0. The complete solution of equation (2.9) is

$$<y^2(t)>_1=\Bigg[<y^2(0)>_1-\Bigg\{\frac{2A\alpha_0}{k}-\frac{2A\alpha\omega}{k^2+\omega^2}+\frac{\alpha_0{}^2}{k^2}-\frac{6\alpha_0\,\alpha k\omega}{(4k^2+\omega^2)}$$

$$-\frac{2\alpha_0\alpha\omega}{k(4k^2+\omega^2)}+\frac{\alpha^2}{2(k^2+\omega^2)}-\frac{\alpha^2\omega k}{2(k^2+\omega^2)^2}+\frac{1}{2k^2}\Bigg\}\Bigg]e^{-2kt}$$

$$+2A\left[\frac{\alpha_0}{A}+\frac{\alpha}{k^2+\omega^2}(k\sin\omega t-\omega\cos\omega t)\right]e^{2kt}$$

$$+\Bigg[\frac{\alpha_0^2}{k^2}+\frac{2\alpha_0\alpha\sin\omega t}{4k^2+\omega^2}-\frac{6\alpha_0\,\alpha k\omega\cos\omega t}{(k^2+\omega^2)(4k^2+\omega^2)}$$

$$+\frac{2\alpha_0\alpha(2k\sin\omega t-\omega\cos\omega t)}{k(4k^2+\omega^2)}+\frac{\alpha^2}{2(k^2+\omega^2)}-\frac{\alpha^2\omega^2 k}{2(k^2+\omega^2)^2}$$

$$+\frac{\alpha^2\omega^2}{2(k^2+\omega^2)}+\frac{1}{2k}\Bigg] \tag{2.10}$$

where

$$A=<y(0)>_1-\frac{\alpha_0}{k}+\frac{\alpha\omega}{k^2+\omega^2}$$

Fluctuation $<y^2(t)>_1$ is exponentially and periodically increasing with time t, and remains periodically finite for large *t*. For the graph of (2.10)we shall take $\alpha_0=0.280,\ \alpha=0.520,$

k = 0.774, ω=0.524, T = 12 months, $< y(0) >_1 = 1.5$, $< y^2(t) >_1 = 1.2$ then $A_1 = -1.2757$.

The graph of equation (2.10) for variance $< y^2(t) >_1$ has been shown time (t) versus $< y^2(t) >_1$.

GOMPERTZIAN GROWTH WITH PERIODICALLY FLUCTUATING DEVELOPMENT PARAMETER IN RANDOM ENVIRONMENT

Let us now suppose that the development parameter, k, varies periodically with time t because of the diurnal or annual changes of environment. If T be the period of ocillation in developmental parameter, then the frequency of oscillation is defined by $1/T$ and angular frequency is defined by $\omega = 2\pi / T$. The simplest possible example of oscillating with angular frequency ω is obtained by assuming for the development parameter, k [7],

$$k(t) = \beta_0 + \beta \cos \omega t \tag{3.1}$$

where

$$\beta_0, \beta > 0 \quad and\, 0 < \beta < 1 \tag{3.2}$$

Here β_0 is the average value of k over a long period of time interval and the condition (3.2) implies the small fluctuation in k relative to β_0. To take account of the influence of randomly fluctuating environment on the development parameter, we add in (3.1) a small perturbing term $\varepsilon(t)$ where properties are given by equation (2.3), then

$$k \equiv k(t) = \beta_0 + \beta \cos \omega t + \varepsilon(t) \tag{3.3}$$

Putting (3.3) in (1.2) we get

$$\begin{aligned} \frac{dy(t)}{dt} &= \alpha_0 - (\beta_0 + \beta \cos \omega t) y(t) + (-y(t))\varepsilon(t) \\ &\equiv \left[a(t) + b(t)y(t)\right] + \left[h(t) + g(t)y(t)\right]\varepsilon(t) \end{aligned} \tag{3.4}$$

$a(t) = \alpha_0, b(t) = -(\beta_0 + \beta \cos \omega t), h(t) = 0, g(t) = 1.$

Now using the nth order moment equation (2.6) and putting $n = 1$ for mean $< y(t) >_2$, we get

$$\frac{d < y(t) >_2}{dt} + (\beta_0 + \beta \cos \omega t) < y(t) >_2 = \alpha_0 \tag{3.5}$$

The solution of the equation (3.5) is

$$A_2 = < y(t) >_2 - A_2 e^{-\beta_0 \frac{\beta}{\omega} \sin \omega t} + a_0 \left\{ \frac{1}{\beta_0} + \frac{\beta(\beta_0 \sin \omega t - \omega \cos \omega t)\omega}{\omega(\beta_0^2 + \omega^2)} \right\} e^{-\frac{\beta}{\omega} \sin \omega t} \tag{3.6}$$

where

$$A_2 = < y(0) >_2 - \alpha_0 \left(\frac{1}{\beta_0} - \frac{\beta\omega}{\beta_1^2 + \omega^2} \right) \tag{3.6a}$$

and $< y(0) >_2$ is t he intial value of $< y(t) >_2$. So that $< y(t) >_2$ is exponetially and periodically increasing with time t, $< y(t) >_2$ tends to be a periodically constant value. For graph we shall take $\alpha_0 = 0.4$, $\beta_0 = 0.65$, $\beta = 0.02$, $\omega = 0.524$, $T = 12$ months, $< y(0) >_2 = 1.5$. The graph of equation (3.6) for average $< y(t) >_2$ has been shown time (t) versus $< y(t) >_2$.

The equation of the second order moment $< y^2(t) >_2$ is given y putting $n = 2$ in equation (2.6), then

$$\frac{d < y^2(t) >_2}{dt} = < y^2(t) >_2 \left[-2(\beta_0 + \beta\cos\omega t) + 1 \right] + 2\alpha_0 < y(t) >_2$$

Using the equation (3.6) in the above equation we get

$$\frac{d < y^2(t) >_2}{dt} + \left\{ 2(\beta_0 + \beta\cos\omega t) - 1 \right\} < y^2(t) >_2$$

$$= 2a_0 \left[A_2 e^{-\beta_0 t - \frac{\beta}{\omega}\sin\omega t} + a_0 \left\{ \frac{1}{\beta_0^2} + \frac{\beta(\beta_0 \sin\omega t - \omega\cos\omega t)}{\omega(\beta_0^2 + \omega^2)} \right\} e^{-\frac{\beta}{\omega}\sin\omega t} \right] \tag{3.7}$$

Final solution of the equation (3.7) for the moment $< y^2(t) >_2$ is given by

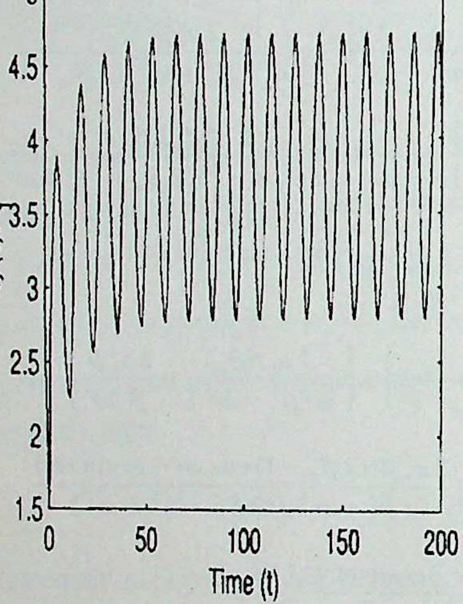
$$
\begin{aligned}
< y^2(t) >_2 = \Big[& < y^2(0) >_2 - \frac{2a_0 A_2}{\beta_0 - 1} + \frac{2a_0 A_2 \beta}{(\beta_0 - 1)^2 + \omega^2} + \frac{2a_0^2}{\beta_0} + \frac{2a_0^2 \beta^2 \beta_0}{2\omega^2(\beta_0^2 + \omega^2)(2\beta_0 - 1} \\
& + \left(\frac{2a_0^2 \beta^2}{\beta_0^2 + \omega^2} + \frac{2a_0^2 \beta}{\beta_0} \right) \frac{1}{(2\beta_0 - 1)^2 + \omega^2} + \frac{2a_0^2 \beta (2\beta_0 - 1)}{(\beta_0^2 + \omega^2)(2\beta_0 - 1)^2 + \omega^2} \\
& + \frac{2a_0^2 \beta_0 \beta^2 (2\beta_0 - 1)}{2\omega^2(\beta_0^2 + \omega^2)(2\beta_0 - 1)^2 + 4\omega^2} \Big] \, e^{-(2\beta_0 - 1)t - \frac{2\beta}{\omega}\sin\omega t} \\
& + \left(\frac{2a_0^2}{\beta_0} + \frac{2a_0^2 \beta^2 \beta_0}{2\omega^2(\beta_0^2 + \omega^2)(2\beta_0 - 1)} \right) + \left(\frac{2a_0 \beta \beta_0}{\omega(\beta_0^2 + \omega^2)} + \frac{2a_0^2 \beta}{\beta_0 \omega} \right). \\
& \Big[\frac{((2\beta_0 - 1)\sin\omega t - \cos\omega t)}{(2\beta_0 - 1)^2 +^2} - \frac{2a_0^2 \beta ((2\beta_0 - 1)\cos\omega t - \omega \sin\omega t)}{(\beta_0^2 + \omega^2)((2\beta_0 - 1)^2 + \omega^2)} \\
& - \frac{2a_0^2 \beta^2 \beta_0 ((2\beta_0 - 1)\cos\omega t - 2\omega \sin\omega t)}{2\omega^2(\beta_0^2 + \omega^2)((2\beta_0 - 1)^2 + 4\omega^2)} \\
& - \frac{2a_0^2 \beta^2 ((2\beta_0 - 1)\sin 2\omega t - 2\omega \cos 2\omega t}{2\omega(\beta_0^2 + \omega^2)((2\beta_0 - 1)^2 + 4\omega^2)} \Big] \, e^{-\frac{2\beta}{\omega}\sin\omega t}
\end{aligned}
\tag{3.8}
$$

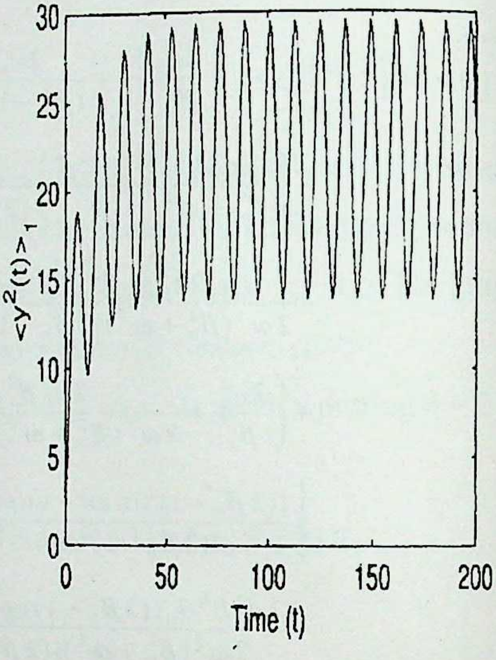
where A_2 is given by (3.6a).

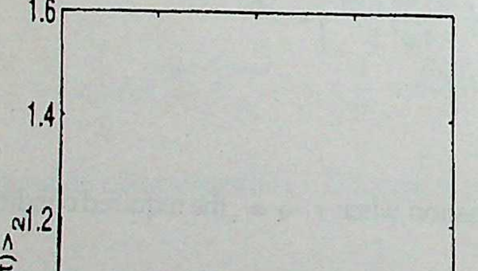
It is clear that for periodically finite fluctuation when $t \to \infty$, the required condition is $\beta_0 > \frac{1}{2}$. When $\beta_0 < \frac{1}{2}$, the fluction is periodically infinite for large t. For the graph we shall take $a_0 = 0.4$, $\beta_0 = 0.65$, $\beta = 0.02$, $\omega = 0.524$, $T = 12$ months, $< y(0) >_2 = 1.5$, $< y^2(0) >_2 = 1.2$, then $A_2 = 0.8906$, $\beta_0 - 1 = 0.35$, $2\beta_0 - 1 = 0.3$. The graph of equation (3.8) for variance $< y^2(t) >_2$ has been shown time (t) versus $< y^2(t) >_2$.

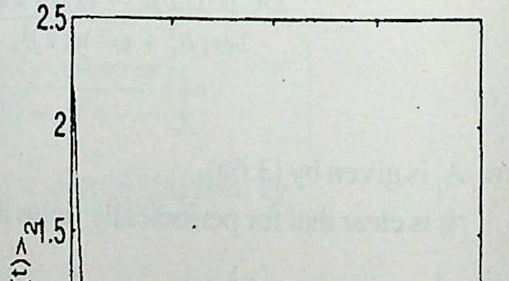
STOCHASTIC RESULTS AND BIOLOGICAL SIGNIFICANCES

Many of the growth phenomena are cyclic or oscillatory in nature. For example, a small tree is planted, begins its growth and total mass-roots, trunk and limbs and leaves continually increases. During the numerous years the tree is growing, its total mass during a year


<y(t)>_1
Time (t)
5
4.5
4
3.5
3
2.5
2
1.5
0
50
100
150
200


<y^2(t)>_1
Time (t)
30
25
20
15
10
5
0
0
50
100
150
200


<y(t)>_2
Time (t)
1.6
1.4
1.2
1
0.8
0.6
0
50
100
150
200


<y^2(t)>_2
Time (t)
2.5
2
1.5
1
0.5
0
0
50
100
150
200

oscillates about an average value: leaves and possibly fruits appear in the storing, the tree is in full foliage during the summer, leaves are shed in the autumn and the basic growth pattern, oscillates in nature which is superimposed on the primary average growth of the tree. Clearly, numerous environmental factors-temperature, sunlight, soil, moistrue, nutrient and like-cause this annual cycle of variations in the growth of the tree [1].

In the present paper we have studied the effects of two factors on the growth of plants for example, a tree. These are the effects of cyclic or periodic variation of environmental parameters (such as growth coefficient and development parameter) and the effect of the randomly fluctuating environment. The actual growth of a tree during a year depends on time-ranging ambient conditions of temperature and rainfall which changes from month to month but are essentially repeated a year later. Then primary productivity of a body of water, its biomass and oxygen generation by photosynthesis, are time-dependent due to diurnal varition of light intensity.

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SEXUAL VARIATIONS IN BLOOD CONSTITUENTS OF FRESHWATER CATFISH, *CLARIAS BATRACHUS*

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ABSTRACT

Sex related variations in twelve blood constituents have been observed in freshwater catfish *Clarias batrachus*, whose population is dwindling at alarming level in and around Lucknow. Interestingly most of the parameters like blood urea, ascorbic acid, total serum proteins, cholesterol, acid and alkaline phosphatase, amylase, SGOT, SOPT and LDH revealed higher levels in females than their respective males, except haemoglobin and serum iron levels. The maximal difference of 33.88% was noticed in serum acid phosphatase while minimal 6.06% in haemoglobin levels between the two sexes. Sex related trends in biochemical blood constituents which may help to understand physiological status offish during breeding, culture and propagation practices have been discussed, comparing to some of those reported for other fishes.

INTRODUCTION

Biochemical characteristics of blood are among the important indices of the status of internal environment of fish organism [6]. Various body and blood constituents of fishes are known to differ in two sexes, since they are related to maturity, breeding cycles and varying growth rate of fish. Haematology no doubt plays an important role in the diagnosis of disease in fish and also in the assessment of the effects of pollution in fish life, unless basic work is done to establish normal haematological ranges this will be difficult [9, 21]. Sex ratio varies considerably from species to species, it differs from one population to another of the same species and may vary from year to year in the same population [26]. Sex related variations in blood constituents have been observed [3 - 8. 11, 13. 14. 20. 31] under specific and varied ecophysiological conditions in different fishes. Following accidental escape of some exotic fishes in Indian streams, population of many native fishes are dwindling at alarming level, Therefore, fishery management should ens me that population of these fishes are sustained at commercially viable level, while meeting societal and economic need.

Clarias batrachus, a prominent catfish is highly popular in India as expensive tablefish, due to its higher nutritional quality as it contains high value of physiological available iron and copper, essential for haemoglobin synthesis and easy digestibility because of low fat content, is facing tough competition in its population due to introduction of Thai mangur *Clarias gariepinus* in its natural habitat [34]. Present paper embodies observations on twelve biochemical blood constituents of freshwater catfish *C. batrachus* in the two sexes, living under natural conditions, which may help to understand physiological status of fish during breeding, culture and propagation

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practices to enrich its decreasing population.

MATERIAL AND METHODS

Live fish were collected from liver Gomti in the vicinity of Lucknow with the help of local fishermen and also brought from suburb areas. Fish were kept in large glass aquaria and allowed to rest for 48 hours. Only mature and healthy fishes were selected for studies of blood parameters round the year. Blood was taken by evering off its caudal end and collected in a clean dry tube, besides in a vial having anticoagulant potassium citrate. Blood ofthe tube was allowed to clot at room temperature for 30 minutes, and then centrifuged at 3000 r.p.m., clear serum decanted in another vial and kept frozen until analyzed. Blood with anticoagulant was analyzed for urea, ascorbic acid and haemoglobin levels following method of atelson [24], Roe [32] and Wintrobe [37] respectively. Serum acid and alkaline phosphatase, oxalacetic (SOOT) and pyruvic transaminase (SOPT) and total serum proteins were determined following the methods given by King and Wootton [17]. Serum lactate dehydrogenase, serum amylase and iron levels were analyzed. using methods given by King [16] and Varley [36] accordingly. Serum cholesterol was estimated using method of Zlatkis et al. [38]. Optical density was measured using Bausch and Lomb spectronic-20 colorimeter. The alimentary canal and other organs of the body along with fresh blood film were also examined for recording general condition and any possible infection.

RESULTS AND DISCUSSION

The results obtained on twelve biochemical blood constituents in two sexes of freshwater catfish *C. batrachus* have been presented in Table-I. Sex related trends were markedly evident in most of the parameters studied here. The maximal difference of33.88% was noticed in serum acid phosphatase while minimal 6.06% in haemoglobin level between the two sexes. Blood urea, ascorbic acid, total serum proteins and alkaline phosphatase levels revealed differences ranging from 15.77% to 16.66% among the two sexes. Significant differences ($P < 0.01$) between the two sexes were seen in serum cholesterol, acid phosphatse, amylase, glutamic pyruvic and oxalacetic transaminase level of the fish. Rest of the parameters like blood urea, ascorbic acid, haemoglobin, total SerLU11 proteins, alkaline phosphatase, iron and lactate dehydrogenase examined here exhibited moderate differences in the two sexes. Interestingly, most of the blood parameters studied here revealed higher levels in females than their respective males except serum iron and haemoglobin levels. Sexual variations have been observed in various biochemical and haematological parameters in many fishes. Various changes in blood parameters indicated the health condition and physiological status of the fish [25] reflecting the relationship of the species to the environment. It has been reported that blood constituents of emales have higher levels than those of male fish under specific and varied ecophysiological conditions [5,11, 12, 18,20,23, 29]. Similar trends were also observed in most of the parameters showing their higher levels in female than their respective males. Proteins serve as structural component, as biocatalyst, as hormone and as a depositor for the characteristic genetic information of species [1]. Higher serum protein levels have been reported in female of *Polyodon spathula* [7] and *Salvelinus fontinalis* [28] than their respective males. Besides total serum proteins, other parameters like serum glutamic oxalacetic transaminase, glutamic pyruvic transaminase and urea, which are closely associated with protein metabolism also revealed significantly higher levels in female specimens. Higher blood urea levels have been reported in females of16 species offreshwater fishes belonging to order cypriniformes [5]. High levels of these parameters can be correlated with their utilization to synchronize increased metabolic and hormonal activities for reproductive preparations. Haemoglobin concentration in blood is known to vary in accordance to life habits, physiological adaptation and intrinsic oxygen affinities [15]. Haemoglobin and iron contents in blood are closely interrelated with stamina and activity of fish [19,30] and both pointed little variations between the two sexes. Such observations in these two blood constituents indicated almost similar physiological status offish, sharing common environment. Joshi [13] also reported higher haemoglobin values in males than the respective female in twelve freshwater fishes, which support present observations. Mulcahy [22] found lower cholesterol levels in male and explained the difference in the levels due to changes in hormone synthesis level associatied with gonadial maturation which is under the influence of an endogenous cycle of gonad that is synchronized with environment and growth rate. Besides, other serum enzymes like lactate dehydrogenase, amylase, acid and alkaline phosphatase also indicated more activityin females than males in present studies. Onishi and murayama [26] also found differences in enzyme levels oftwo sexes of salmonid fishes. The enzyme levels in two sexes have been reported to vary in fishes due to variations in size [27], different maturation time [2, 33] and longer life offemale [10]. Slicher [35] suggested that certain hormonal activity is more or less responsible for such intra species sex related variations. During phases of life on account of sexual activity and related behaviour in either sex reflect its impact on the blood composition of fish. These parameters can also be used to assess the status of fish during culture practices and monitoring hazardous effects of aquatic pollution, leading to fast depleting population of fish in reference to biodiversity conservation.

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Table-1: Blood constituents in the two sexes of *Clarias batrachus*

Parameter	**Mean ± Std. Deviation** **Male (48)***	 **Female(56)***
1. Blood urea (mg/100ml)	5.64 ± 1.76	6.72 ± 1.88
2. Blood Ascorbic acid (mg/100ml)	0.05 ± 0.01	0.06 ± 0.02
3. Haemoglobin (gms/100ml)	10.86 ± 2.82	10.14 ± 2.14
4. Total Serum Proteins (gms/100ml)	5.24 ± 1.80	6.16 ± 2.10
5. Serum Cholesterol (mg/100ml)	386.00 ± 60.42	470.0±88.20
6. Serum Acid Phosphatase (KA units/100ml)	24.18 ± 6.90	35.06 ± 8.10
7. Serum Alkaline Phosphatase (KA units/100ml)	4.70 ± 1.80	5.58 ± 1.72
8. Serum Amylase (units/100ml)	148.00 ± 48.40	210.0±70.10
9. SGOT (units/ml)	28.57 ± 6.70	35.80 ± 7.56
10. SGPT (units/ml)	16.92 ± 4.90	24.66 ± 6.18
11. Serum Iron (g/100ml)	19.47 ± 8.00	17.56 ± 7.50
12. Serum LDH (units/100ml)	479.40 ± 88.00	546.06 ±97.58

*No. of observations Weight range- 260-300 gms

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COMMON FIXED POINT OF A PAIR OF QUASI-NONEXPANSIVE AND ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0)-SPACES

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ABSTRACT

We obtain a fixed point theorem for a quasi-nonexpansive mapping and an asymptotically nonexpansive mapping in the framework of a complete CAT(0) spaces .

Keywords: Quasi-nonexpansive nonexpansive mapping; asymptotically nonexpansive mapping; fixed point.

MSC (2000) 47H10, 54H25

INTRODUCTION

In 1972, K. Goebel and W.A.Kirk introduced a new class of mappings namely asymptotically nonexpansive, which have been studied mostly in the context of uniformly convex normed spaces. On the other hand, in 1967 Diaz and Metcalf [2] introduced the concept.of quasi-nonexpansive mappings. Both classes of mappings have been studied intensively under various settings. Our contribuition here is to establish a new result about existence of a common fixed point between two mappings S and T, where S is quasi-nonexpansive and T is an asymptotically expansive mapping, in the context of CAT(0) spaces.

We remark that in 1972, Goebel and Kirk proved the following generalization of the famous Browder-Goehed-Kirk fixed point theorem for nonexpansive mappings.

Proposition 1.1. (see [3]) Nonempty closed convex and bounded subsets of uniformly convex Banach spaces have the FPP for asymptotically nonexpansive mappings.

In 2004, W.A. Kirk obtained a similar result for a CAT(0) space.

Proposition 1.2 (see [6]) Nonempty closed convex and bounded subsets of complete CAT(0) spaces have the fixed point property (FPP) for asymptotically nonexpansive mappings.

Proposition 1.3 (see [4]) Let X be a convex subset of a uniformly convex metric space and $S: X \to X$ a quasi-nonexpansive mapping whose fixed point set is nonempty. Then

Fix (S) is closed convex.

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At this point for the sake of completeness, we recall some basic definitions and introduce some notations which will be used in this note.

Notation 1.4. If X is a nonempty and $S:X\rightarrow X$, we denote by Fix (S) the collection of all $p \in X$, such that $S(p) = p$

Remark. 1.5 (see [6], [1]). A CAT(0)-space is a specific type of metric space (see [1] for a detailed treatment).

Definition 1.6 (see [5]). Let (M,d) be a metric space. A mapping $T: M\rightarrow M$ is said to be asymptotically nonexpansive if there exists a sequence of positive numbers $\{k_n\}$, with $\lim k_n = 1$, when $n \rightarrow \infty$, such that

$$d(T_n(x), T_n(y)) \leq k_n d(x, y) \text{ for any } x, y \in M, \text{ and } n = 1, 2, \ldots$$

Definition 1.7 (see [4]). Let (X, d) be a metric space. A mapping $f: X\rightarrow X$ is said to be quasi-nonexpansive if $d(f(x), p) \leq d(x,p)$ for all $x \in X$ and $p \in$ Fix (f).

Definition 1.8 (see [4]). Let (X, d) be a metric space and $I = [0,1]$. A mapping $W:X\times X\times I\rightarrow X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X\times X\times I$ and $z \in X$, $d(z, W(x, y, \lambda) \leq \lambda\, d(z,x) + (1-\lambda)\, d(z, y)$.

Definition 1.9 (see [4]). A metric space (X, d) together with a convex structure W is called a convex metric space, which will be denoted by (X, d, W).

Definition 1.10 (see [8]). A convex metric space (X, d, W) is said to be uniformly convex if for any $\varepsilon > 0$, there exists $\mathrm{S} = \mathrm{S}(\varepsilon) > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z,y) \geq r$ and $d(x,y) \geq r\varepsilon$, $d(z, W(x, y, \frac{1}{2})) \leq r(I - S)$.

Remark 1.11 (see [4]). It is easy to conclude that

a) uniformly convex Banach spaces are uniformly convex metric spaces.

b) CAT(0) spaces are also uniformly convex.

Definition 1.12 (see [8]}. Let S be a subset of a convex metric space. S is said be convex if for every $x \in S$, $y \in S$ and $\lambda \in [0,1]$ we have $W(x, y, \lambda) \in S$.

THE MAIN RESULT

The main result of our work goes as follows

Theorem 2.1. Let X be a convex and bounded subset of a complete CAT(0)-space. Let $S:X\rightarrow X$ a quasi-nonexpansive mapping whose fixed point set is nonempty. Suppose that

$T:X\to X$ is an asymptotically nonexpansive mapping. If one of the following conditions holds,

a) $SoT = ToS$ $$(2,1)$$

or

b) $\|u-T(u)\| \leq N(\|u-S(u)\|,\| T(u) - (SoT)(u) \|)$ $$(2,2)$$

for every $u \in X$, where $N: R^{+}\times R^{+}\to R^{+}$ is a function such that

$N(0, t) < t$, if $t > 0$

or

c) $\| (SoT)(u) - T(u) \| \leq M (\|(ToS)(u)-T(u)\|)$ $$(2,3)$$

for each $u \in X$, where $M: R^{+}\to R^{+}$ is a function such that $M(s) < s$, if $s > 0$; then $\mathrm{Fix}(S)\cap \mathrm{Fix}(T) \neq \emptyset$, that is, S and T have a common fixed point in X.

Proof: From [6], we observe that the metric space CAT(0) is uniformly convex. Further, Fix (S) $\subset X$ and X is supposed bounded. So, Fix(S) is also bounded and, by Proposition 1.3, Fix(S) is nonempty closed and convex subset of X.

If $SoT = ToS$, it is easy to see that T: Fix (S) $\to$ Fix(S), because in this case $u \in$ Fix (S) implies $u \in X$ and $S(u) = u$ and therefore $TS(u) = ST(u) =.T(u)$. If S and T satisfies (2.2.) or (2.3.), and u Fix(S) and $T(u)$ is not in Fix(S), we obtain imediately a contradiction, since we have $N(0, t) < t$, if $t > 0$ and $M(s) < s$, if $s > 0$. Hence under the hypothesis (2.2) or (2.3), we get $T(\mathrm{Fix}(S)) \in$ Fix (S). Therefore, since T: Fix(s) $\to$ Fix(S) also is asymptotically nonexpansive mapping, we note that all considerations for appliying Proposition 1.2 are fullfilled.

Hence, there exists at least a $p\in$ Fix(S) such that $T(p) = p$, that is, $\mathrm{Fix}(S)\cap \mathrm{Fix}(T) \neq \emptyset$. This completes the proof.

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GRACEFUL GRAPHS FROM HAIRY CYCLES

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ABSTRACT

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In this paper graceful labelings of subdivision of hairy cycles, subdivision of the pendent edges attached to each vertex of the hariy cycle $C_n \odot 1K_1$ and graceful labeling of two isomorphic copies of the hairy cycle $C_n \odot 1K_1$ by merging each pendent vertex of one copy hairy cycle $C_n \odot 1K_1$ to its isomorphic copy have been studided.

Keywords: graceful labeling, hairy cycle, unicyclic graph, corona of graph, subdivision of graphs.

INTRODUCTION

A graph with a graceful labeling is called a graceful graph. The concept of graceful labeling was first introduced by Ringel-Kotzig-Rosa [4], [5], [6] and [7]. By a graceful labeling of a graph G with q edges, we mean an injection $f: V(G) \rightarrow \{0,1,2.........,q\}$ such that $\{|f(u)-f(v)| : (u,v) \in E(G)\} = \{1,2,3,..........., q\}$. Truszcy nski [8] conjectured that all the unicyclic graphs (graphs that have only one cycle) except C_m where $m \equiv$ 1 or 2 (mod 4), are graceful. A caterpillar is a tree with the property that the removal of all the pendent vertices reduces it to a path. A unicyclic graph G other than a cycle with the property that the removal of any edge from the cycle reduce G to a caterpillar is called hariy cycle. Barrientos [1], [2] proved that all hariy cycles are graceful. A lobstar is a tree with the property that the removal of all pendent vertices reduces it to a caterpillar

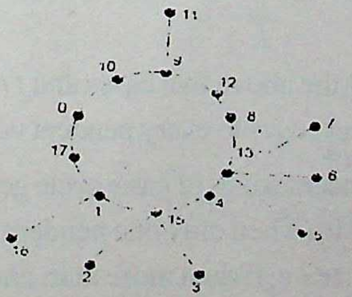

Fig 1: A hairy cycle with a graceful labeling.

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Fruchut and Haray [3] presented a new operation between two graphs, namely, the corona of two graphs. Given two graphs G and H, the corona of G with H, denoted by $G \odot H$ is a graph with $V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i)$ and $E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} E(H_i) \cup \{(i, u_i) : i \in V(G)$ and $u_i \in V(H_i)\}$. In other words, a corona graph is obtained from graphs G and H by taking one copy of G with order say p, and p copies of H, and then joining the K^{th} vertex of G to every in the K^{th} copy of H by an edge. The corona of C_n and mK_1, i.e $C_n \odot mk_1$ is the example of hairy cycle.

MOTIVATION

Motivated by the results in [1], an algorithm has been developed to construct graceful graphs by attaching pendent edges to each pendent vertex of the hairy c ycle $C_n \odot 1k_1$.

Proposed Algroithm: Let G be a graph obtained from the hairy cycle $C_n \odot 1k_1$ by attaching exactly one pendent edge to every pendent vertex of the hairy cycle $C_n \odot 1k_1$. Let $|E(G)| = m$ We observe that the distance of every pendent vertex in G from the unicycle is exactly two. Furthermore, the removal of any edge of the unicycle in G reduces it to a lobster. By [1], there exists a gracful labeling f of $C_n \odot 1k_1$ in which a pendent vertex gets the label 0. We know that a hairy cycle is bipartite graph and so G is bipartite. Let $\{A, B\}$ be the bipartition of the vertices of G

Step I:- If u_1 be the pendent vertex of the hairy cycle at which one pndent edge $u_1v_1 \in E(G)$ is attached and let $f(u_1) = 0, f(v_1) =$ m and $f(v_2) = m-1$. If more than one pendent edges are attached to each pendent vertices of hairy cycle, then $f(u_1) = 0, f(v_1) =$ m-1 and $f(v_2) =$ m.

Step II:- If exactly one pendent edge is attached to one pendent vetex of hairy cycle, then $f(u_2)$- 1 otherwise $f(u_2)$ = 2.

Step III:- Give $f(v_3) =$ m - 2 for the above two cases and $f(u_3) = 1$ for the graph G obtained by attaching two or more pendent edges to every pendent vertices of hairy cycle.

Step IV:- As soon as the two pendent edges of hariy cycle get lables differently, then give the lable to the vetex u_1 adjacent to v_1 (when only one pendent edge is attached to the pendent vertex of hairy cycle) or the vertex v_2 (when more than one pendent edges attached to the pendent vertices of hairy cycle) such that $u_1v_1 \in E(G)$ of $u_1v_2 \in E(G)$ get the label one less than edge lable $u_2v_3 \in E(G)$

Step V:- Give the label to the vertex v_4 such that the edge $u_2v_4 \in E(G)$ gets the ladel one less than the edge of $u_iv_1 \in E(G)$ or $u_iv_2 \in E(G)$.

Step VI:- Increase the label of u's and decrease the label of v's until the differently all the edges and vetices of G get the distinct label and after that when all the edges and vertices have distinct label then terminate the algorithm.

Theoerm1: Subdivision of cycle in all graceful hairy cycles are graceful hairy cycles.

Proof : It is obvious that by subdivision of cycle in hairy remains a hairy cycle. So, by subdivision of cycle is graceful hairy cycle is graceful hairy cycle.

Theorem 2: Subdivision of the pendent edges attached to each vertex of graceful hairy cycle is a graceful graph.

Proof : The figure 9 shown below proves the theorem.

Theorem3 : Graph obtained from two isomorphic copies of $C_n \odot 1k_1$ to its isomorphic copy of the second copy of $C_n \odot 1k_1$.

Proof : The figure 10 proves the theorem.

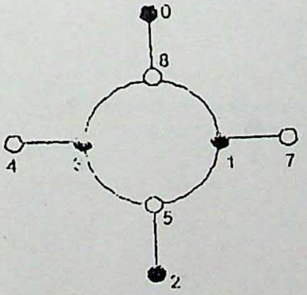


Fig. 2: Hairy Cycle

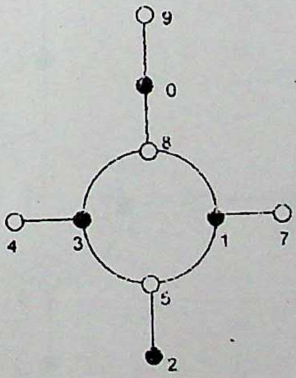


Fig. 3: Hairy Cycle with one pendent edge attached

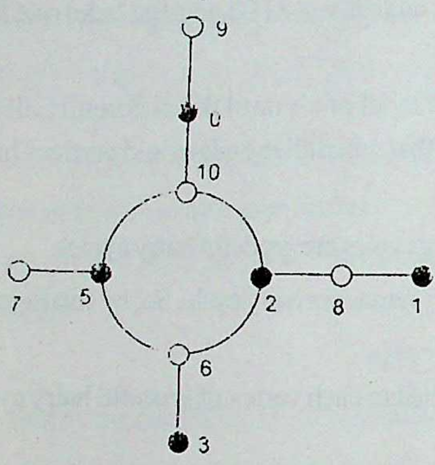

9
0
10
7
5
2
8
1
6
3

Fig. 4: Hairy Cycle with two pendent edges attached

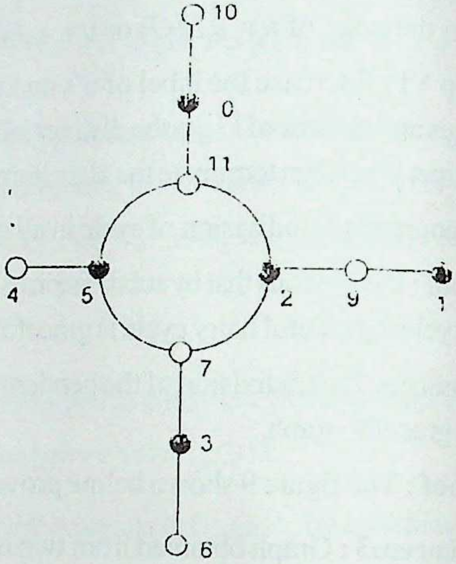

10
0
11
4
5
2
9
1
7
3
6

Fig. 5: Hairy Cycle with three pendent edges attached

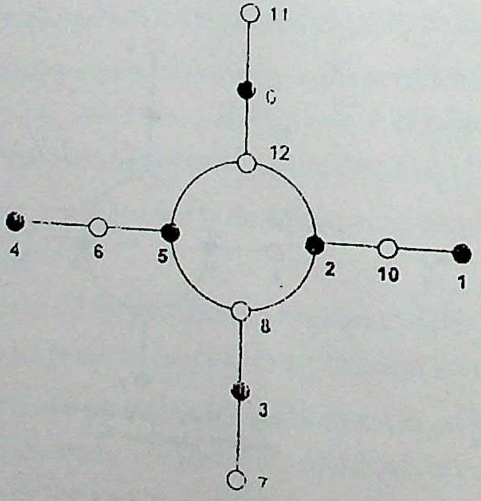

11
0
12
4
6
5
2
10
1
8
3
7

Fig. 6: Hairy Cycle with four pendent edges attached

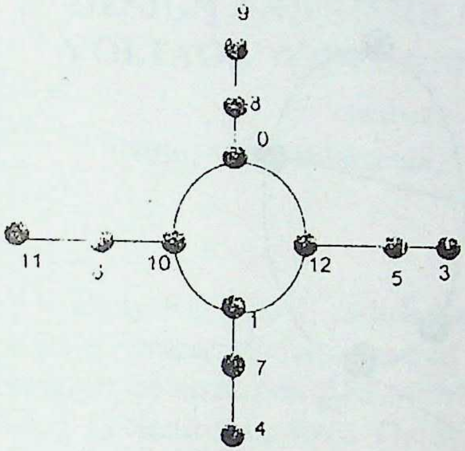
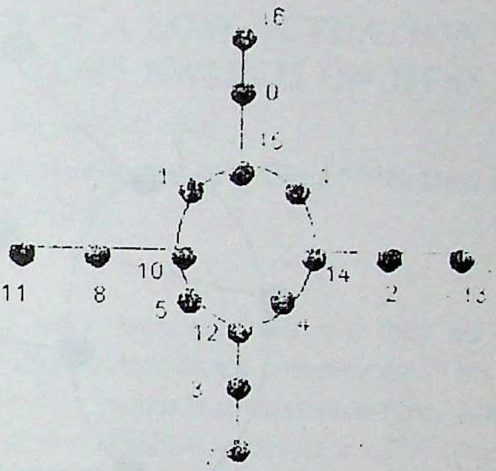

9
8
0
11
6
10
12
5
3
1
7
4

Fig. 7: Graceful hairy cycle with pendent edges


16
0
15
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11
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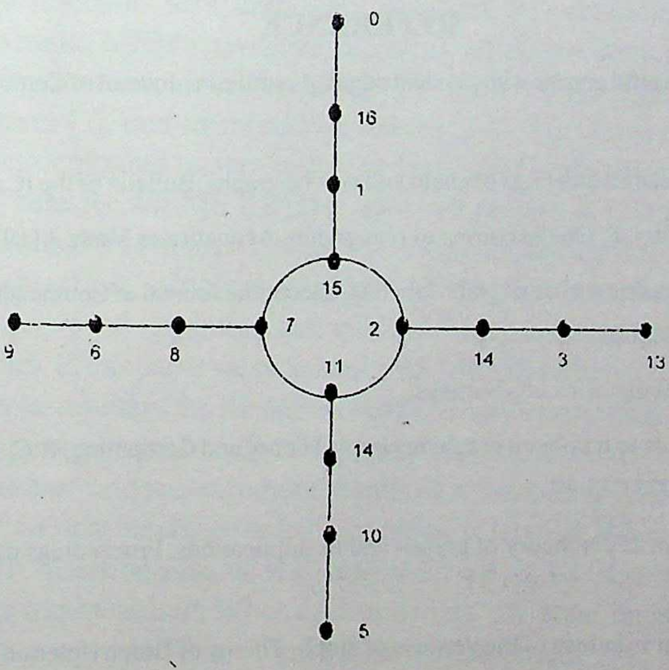
12
4
2
13
3

Fig. 8: Graceful hairy cycle with pendent edges obtained by Subdivision of cycle in graceful hairy cycle


0
16
1
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7
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14
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13
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10
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Fig. 9 . Subdivision of pendent edges attached to each vertex of hairy cycle

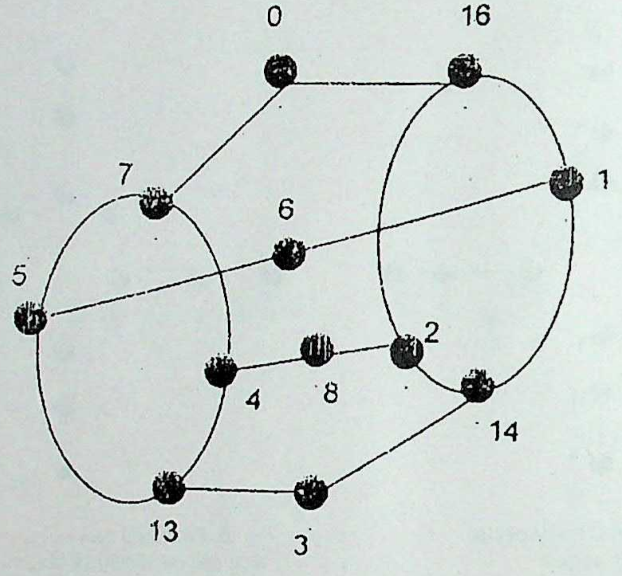

Fig. 10

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DESIGN AND SIMULATION OF A LOW ACTUATION VOLTAGE WIDEBAND RF MEMS SWITCH ON HFSS

(Received 04.10.2009, Revised 07.12.2009)

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Many RF MEMS switch topologies have been reported and they all show superior RF characteristics compared to semiconductors based components. This work investigates the simulation of RF MEMS switch. The switch is a capacitive type, which is actuated by electrostatic force. The structure of the switch consists of a CPW (coplanar waveguide) transmission lines and a suspended membrane. The switch has an insertion loss of less than 0.25dB and an isolation of more than 45dB over the frequency range from 1 to 10GHz. Simulated results show that the pull in voltage of the switch is about 10 volts.

INTRODUCTION

The low insertion loss and high isolation RF characteristics over a wide frequency range make MEMS switches the most attractive devices for reconfigurable antenna and integrated circuit applications [1]. Most of RF MEMS switches utilize rotary [5], cantilever [7], and membrane structures [2-4]. They have demonstrated superior RF characteristics compared to active-device-base solid state switches such as FET and diode switches. These RF MEMS devices, however, require very high actuation voltages (usually 30-50 volts). The high voltage operation is far beyond standard monolithic microwave IC operation, which is around 5 Volts DC biased operation. There is a crucial need for improved apparatus and method which address this drawback of known RF MEMS switches. In this paper we proposed a RF MEMS switch for low voltage operation [6]. The experimental results of the RF MEMS switch show a promising low voltage operation and excellent RF performance over the frequency band from 1GHz to10 GHz. This paper presents the *simulation* and measurement results of a "cantilever" RF MEMS switch. The simulated results show that the device switching voltage is 10 volts. When the device is operated in an 'on' state, an insertion loss of less than 0.25 dB at 10 GHz with a return loss of -36 dB at 10 GHz was measured. When operated in its 'off' state, an isolation of better than 45 dB over the frequency band from 1 to 10GHz was achieved.

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DEVICE STRUCTURE

The schematic cross-sectional view of the RF MEMS switch is shown in Fig. 1. The RF signals are guided by a coplanar waveguide (CPW) structure, which is composed of a signal line and ground planes on both sides. The CPW in this switch is disconnected in the middle, when on applying the voltage over the membrane, the membrane can be used to join the two disconnected ends of the CPW and hence make the switch ON. The actuation voltage provides an electrostatic force to make the membrane move up and down. When voltage is applied on the membrane, the voltage on the input port of the CPW will directly pass to the output port through the membrane and this forms the "on" condition of the switch. On the other hand when no voltage is applied to the membrane, no electrostatic force is developed between the membrane and the CPW as a result of this, the voltage through the CPW can't get transferred to the output port and this forms "off" condition of the switch.

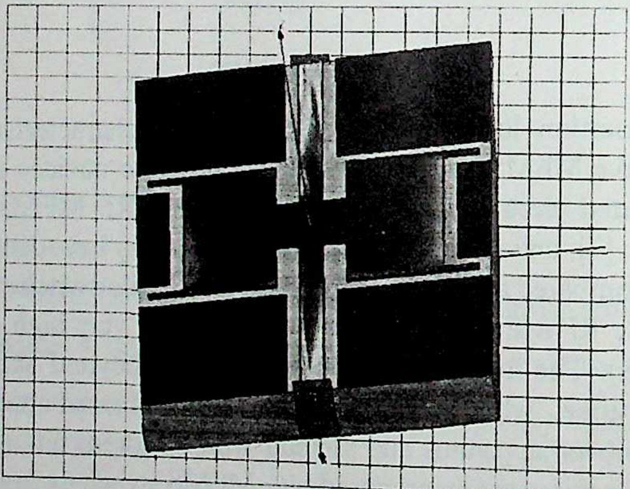


Fig. 1: The schematic cross-sectional side view of RF MEMS switch with hinge configuration

DESIGN PARAMETERS OF THE SWITCH

Coplanar waveguide- Substrate- Rogers RO 4232(tm), permittivity- 3.2, dimensions 420*300 μm², height- 50μm.

Metal line- Copper, Dimensions 420*20 μm², Height -2.1 μm, Slot -12 μm.

Ground plane- Copper, dimensions- 140*128 μm², height-2.1μm

Switch - Central contact- gold, dimensions 56*60 μm^2, height 1 μm

Side cantilevers- Gold, dimensions -150*80 μm^2, height -1 μm

Anchor- Gold, dimension- 4*4 μm^2, height -1 μm.

Joint between cantilever and anchor- Gold, dimensions- 4*30 μm^2, height-1 μm.

Cut in the metal line- 40*20 μm^2

Gap between switch and metal line - 1 μm.

Height of Radiation box - 1mm.

Structures of the switch in on and off positions are shown in Fig.2 -3.

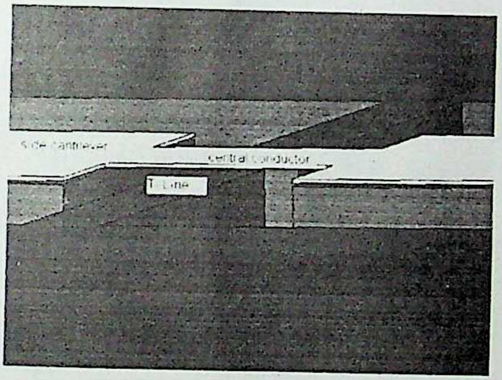


Fig. 2 Switch in OFF state

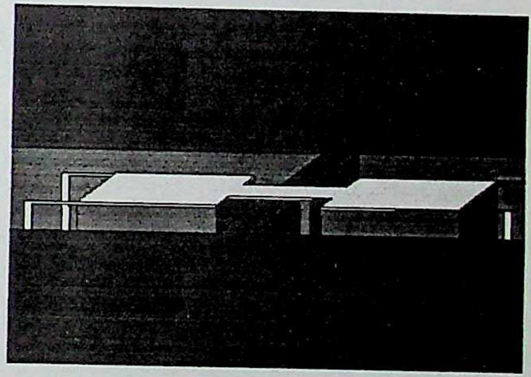
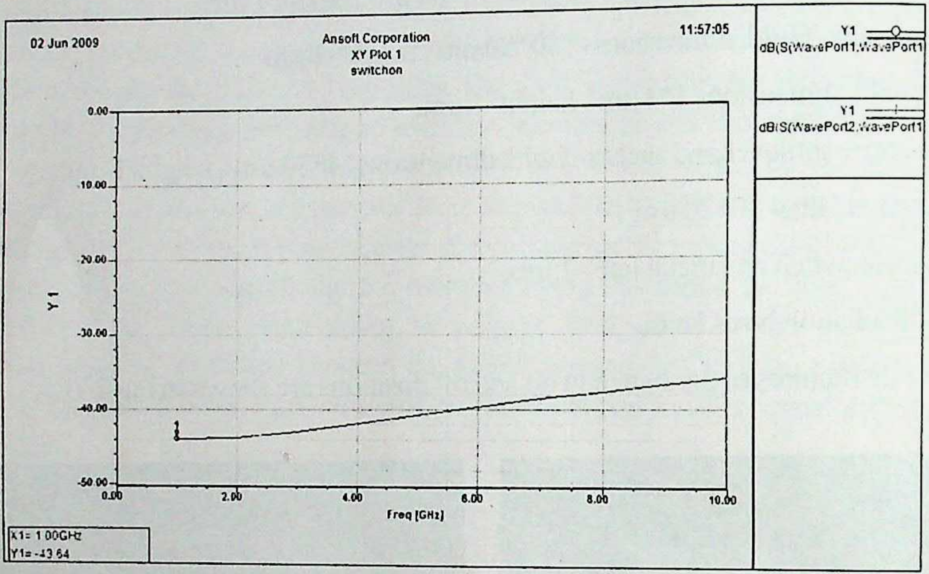


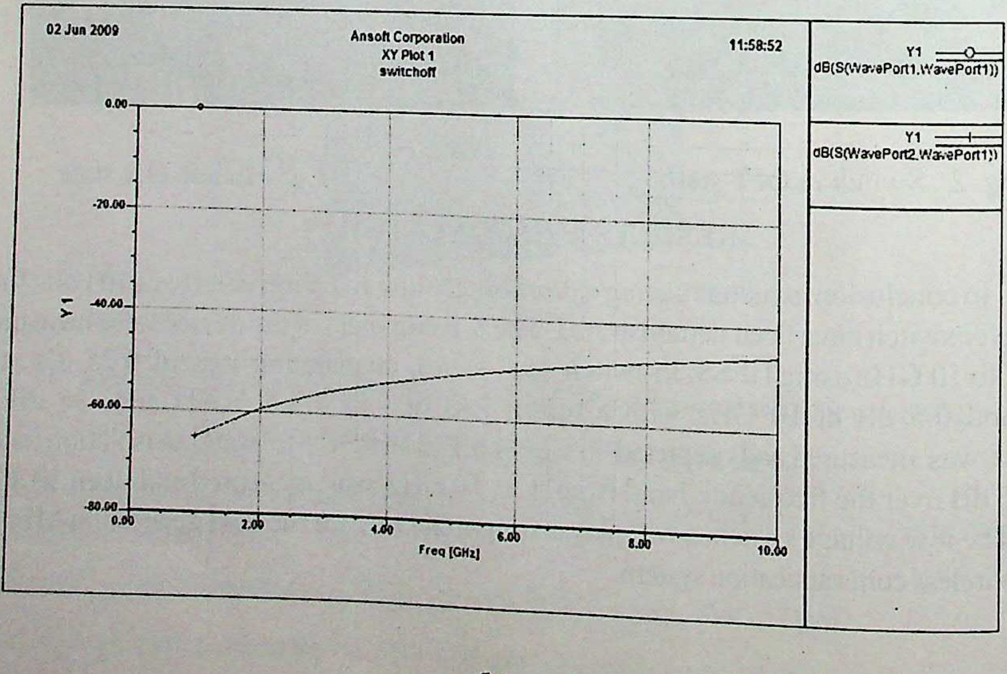
Fig.3 Switch in ON state

RESULTS AND CONCLUSION

In conclusion, structures using a promisingly low actuation voltage of 10 volts for a cantilever switch have been demonstrated. The S-parameters of the device were measured from 1 to 10 GHz using HFSS. In switch 'on' state), an insertion loss of 0.25 dB at 1 GHz and 0.5 dB at 10 GHz with a return loss of - 44 dB at 1GHz and -36 dB at 10 GHz was measured and depicted in Fig. 4 (a). In switch "off" state, an isolation better than 45 dB over the frequency band from 1 to 10 GHz was measured and seen in Fig. 4(b) . The low voltage switches will play an important role for the next generation MEMS based wireless communication system.



(a)



(b)

Fig. 4: The RF performance of MEMS switch for (a) 'switch on' state and (b) 'switch off state.

Acknowledgement: The authors are thankful to DEAL, Dehradun, for the technical and lab support they provided.

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

खण्ड 23 अंक (1–2) 2009

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हस्ताक्षर
प्रो0 ए. के. चोपड़ा
कुलसचिव

Journal of Natural & Physical Sciences, Vol. 23 (1-2) (2009) 91-98

RADIATION DUE TO STEPPED LEADER IN HF RANGE DURING GROUND DISCHARGE

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(Received 07.10.2009, Revised 09.02.2010)

ABSTRACT

Stepped leader emits radiation in whole of radio spectrum from extremely low frequency (ELF) to microwave region and beyond up to visible region. In this paper, the electric field from stepped leader in higher frequency range by positive corona streamer system have been calculated and compared with that in 1 MHz – 20 MHz range as observed by Weidman and Krider [15]. Observed values are also plotted along with calculated values for comparison. It is seen that calculated values are in very good agreement with observations.

Key Words: Stepped leader, Corona steamer, Electromagnetic radiation.

INTRODUCTION

The usual cloud-to-ground lightning discharge is initiated by a negatively charged stepped leader that propagates from cloud-to-ground. During cloud-to-cloud discharge and cloud-to-ground discharge, electromagnetic radiations are emitted which cover a wide frequency range from extremely low frequency [(ELF) < 3 kHz] to ultra violet (UV) region. Even before the actual discharge some radiation is observed due to pre-discharge breakdown known as preliminary breakdown, though the radiation is only in very high frequency [VHF(30 - 300 MHz)] range. In very low frequency [VLF (3 - 30 kHz)] range the observed power is accounted for the radiation from a variable dipole constituted by the propagating lightning channel and its image in the finitely conducting ground [4, 5]. In ELF region the emitted power is accounted by lateral corona current flowing towards return stroke channel within the stepped leader sheath [9, 10]. The ultra high frequency [UHF(0.3 - 3 GHz)] power attributed to the bremsstrahlung radiation arising due to acceleration or deceleration of electrons in their orbit around the ions or that of atom due to coulomb field interaction in the lightning channel [11, 12]. The physical processes behind the radiation in the microwave region above UHF are not yet explained.

The mechanism applicable in ELF, VLF and UHF region [5, 12] are not applicable or do not give sufficient radiation in the large region from low frequency (LF) to VHF (30 kHz-300 MHz) range. The stepped leader is well known to emit the radiation in LF to VHF and microwave regions [3]. Mechanism responsible for this radiation is also not known. To the authors' knowledge, no work has been done on the mechanism for radiation in this region [14] except that by our group [3,6-8]. Pathak and Rajput [6] have calculated raditation from

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stepped leader but they have taken velocity of stepped leader as 5x10^7 m/s. However, according to Cooray [1] the velocity of stepped leader does not exceed 5.7x10^5 m/s. Here we revise these calculation with modified value of the parameters.

THEORY

It is known that the cloud to ground lightning consists of mainly two types of strokes. It starts with stepped leader i.e. column of partially ionized channel of radius of few meters propagating from cloud to ground. On reaching near ground this leader triggers return stroke i.e. a fast moving fully ionized thin column of radius of about a centimeter. We assume the stepped leader to consist of a number of positive corona streamers propagating simultaneously. All such streamers give away radiations. Radiation from single streamer is calculated by Pathak [3]. Propagation of streamer for time t in z - direction produces a dipole moment [3]

$$\vec{p} = \frac{Q}{4} z\hat{k}$$

where Q is charge on the tip of corona streamer. For corona streamers in pre-discharge process Pathak [3] assumed exponential growth for small distance and z^2- growth afterwards. But the stepped leader is more intense than pre-discharge process. Therefore, it can be assumed that growth is z^2 dependent from starting i.e. $Q = Q_0 z^2$. With this dipole moment becomes

$$\vec{p} = \frac{Q_0 v_s^3 t^3}{4}\hat{k} \tag{1}$$

where v_s is the velocity of corona streamer and Q_0 is the initial charge on the streamer tip or head. The radiated field at a distance r from a dipole is given by

$$\xi = \frac{1}{4\pi\varepsilon_0 c^2 r}\frac{d^2 p}{dt^2} \tag{2}$$

where, c is the speed of light. Now from equation (1)

$$\frac{d^2 p}{dt^2} = \frac{3}{2} Q_0 v_s^3 t \tag{3}$$

On substituting equation (3) in equation (2), one gets

$$\xi = \frac{1}{4\pi\varepsilon_0 c^2 r}\,\frac{3}{2}\,Q_0 v_s^3 t \tag{4}$$

If number density of corona streamer is *n*, the expression of electric field at a distance *D* from foot of the channel due to a disc of thickness *dx* at height (*H-x*) of the channel from ground (Fig. 1) is given by

$$dE = \pi a^2 n \xi dx = \frac{1}{4\pi\varepsilon_0 c^2}\,\frac{3}{2}\,\frac{Q_0 n \pi a^2 v_s^3 t dx}{\sqrt{(H-x)^2 + D^2}} \tag{5}$$

where, *a* is the radius of leader channel. Electric fields, due to whole channel are the integral of equation (5) with *v* being the velocity of stepped leader, i.e.

$$E = \frac{1}{4\pi\varepsilon_0 c^2}\,\frac{3}{2}\, n\pi a^2 v_s^3 Q_0 v \int_0^t \frac{t dt}{\sqrt{(H-x)^2 + D^2}}$$

$$= E_0 I$$

where,

$$E_0 = \frac{1}{4\pi\varepsilon_0 c^2}\,\frac{3}{2}\, n\pi a^2 v_s^3 Q_0 v$$

and $I = \int_0^t \frac{t dt}{\sqrt{(H-x)^2 + D^2}}$

On solving the integral, we get

$$I = \frac{1}{v^2}\left[\sqrt{(H-vt)^2 + D^2} - \sqrt{H^2 + D^2}\right] - \frac{H}{v^2}\left[\ln\left\{(H-vt) + \sqrt{(H-vt)^2 + D^2}\right\} - \ln\left(H + \sqrt{H^2 + D^2}\right)\right]$$

CALCULATION AND RESULTS

We idealize a section of the stepped leader channel of height *H* (of range 3 to 4 km) as a straight vertical line above perfectly conducting plane (the earth) as shown in Fig.1. Values of parameters are mostly taken from Uman [14] unless the reference is quoted for some parameter. The radius of leader channel is of few meters in the range 1 to 10 m. A most frequently observed (mfo) value of 3 m can be taken for calculations. The stepped leader moves downward in a series of discontinuous luminous steps that ranges in length from 10 to 200 m (mfo value of 50 m can be taken for calculations) and that are separated by pause time of 40 to 100 μsec. Longer paused times are generally followed by longer steps. It was observed that there is a mean time of 50 μsec between steps far above the ground. The step luminosity is of the order of 1 μsec or less. A fully developed stepped leader lowers up to 10 or more coulombs negative cloud charge towards ground in tens of milliseconds. We have calculated the electric field of leader steps for a discharge over 50 km distant, which is denoted by *D* in above formulation. The velocity of corona streamer is about 5×10^5 m/s [3]. The velocity of stepped leader is 5.7×10^5 m/s [3]. The initial charge on the tip of corona streamer (Q_0), is taken 4.61×10^{-7} C [3].

With these values, the electric field from stepped leader of 4 km height at a distance of 50 km is calculated. As the observations of Weidman and Krider [15] are in frequency (*f*) domain, we need to convert our result also in frequency domain. For this an expression of *E* in time (*t*) domain is needed. To get such an expression the values of electric field are first plotted with time. While plotting these values the pause time of the steps is added to obtain the real time after start of stepped leader (Table.1). Though, this is not the time of propagation of leader which is simply total length of leader up to that instant divided by velocity. The values of $E(t)$ are fitted to a parabolic curve $A + Bt + Ct^2$ and the values of constants *A*, *B* and *C* are evaluated. We found that the value of constant *C* is almost zero. Then by Fourier transform $E(t)$ is converted to $E(f)$ and 20log $E(f)$ is evaluated and plotted with log (*f*). Observed values of Weidman and Krider [15] are also plotted along with calculated values for comparison in graph. Though the observations are available in the range 1-20 MHz, the calculations are made from 1 MHz to 300 MHz just to show the variation of emitted field for wider range.

DISCUSSION

As mentioned above the variation of computed electric field of radiation on the basis of positive corona streamer in higher frequency range from leader channel is shown in graph (Fig.2) along with the observed field of Weidman and Krider [15]. It is clear that there is very good agreement between them. The fields were calculated for discharge during stepped leader at a distance of 50 km as the observations of Weidman and Krider [15] are at same distance. The height of leader channel is taken 4 km. as the cloud-to-ground lightning is more frequent in convective clouds, which generally starts at the heights of many kilometers varying from 2 km to 10 km or more. The typical value of 4 km is taken for the calculations. The mfo value of radius of leader channel is taken as 3 m. We further considered that the stepped leader moves downward in a series of discontinuous luminous steps of mfo value of length 50 m. There is a pause time between steps.

The mfo value for this is 50 μsec. The stepped leader consist of a number of positive corona streamer propagating simultaneously, all these corona streamers gives radiation. The number of corona streamers values is taken 20 m^{-3} during stepped leader, for these calculations. The average value of the velocity of corona streamer taken for calculation is 5×10^5m/s [3].

The velocity of stepped leader is 5.7×10^5 m/s [1], denoted by v in above formulation.

Table: 1. Electric field $E(t)$ at different times after initiation of Stepped leader

Propagation Time (μs)	Real Time t (μs)	Electric Field $E(t)$ in V/m
1	1	0.00002791086
2	53	0.00784040000
3	106	0.03136260400
4	107	0.71458560000
5	215	1.29034360000
6	271	2.05013000000
7	328	3.00334320000
8	386	4.15955200000
9	445	5.52849200000
10	505	7.12007200000
11	566	8.94436800000
12	628	9.81162400000
13	691	13.33225200000
14	755	15.91684000000
15	820	18.77613600000
16	886	21.92107200000
17	953	25.36272800000
18	1021	29.11237600000
19	1090	35.81440000000
20	1160	37.58152400000
21	1231	42.04440000000
22	1303	47.42000000000
23	1376	52.88644000000
24	1450	58.73000000000
25	1525	64.96512000000
26	1601	71.60444000000
27	1678	78.66072000000
28	1756	86.14692000000
29	1835	94.36160000000
30	1915	102.46176000000
31	1996	111.31720000000
32	2078	120.65604000000
33	2161	130.49216000000
34	2245	140.83952000000
35	2330	151.71220000000

36	2416	163.12460000000
37	2503	175.09112000000
38	2591	187.62648000000
39	2680	200.74544000000
40	2770	214.46304000000
41	2861	228.79440000000
42	2953	243.75484000000
43	3046	259.35988000000
44	3140	275.62516000000
45	3235	292.56652000000
46	3331	310.19996000000
47	3428	328.54164000000
48	3526	347.60788000000
49	3625	367.41524000000
50	3725	387.98028000000
51	3826	409.32000000000
52	3928	431.47200000000
53	4031	454.39120000000
54	4135	478.15720000000
55	4240	502.76680000000
56	4346	528.23800000000
57	4453	554.65880000000
58	4561	581.83520000000
59	4670	609.99760000000
60	4780	639.09360000000
61	4891	669.14200000000
62	5003	700.16080000000
63	5116	732.16920000000
64	5230	765.18640000000
65	5345	799.23160000000
66	5461	834.32320000000
67	5578	870.48200000000
68	5696	907.72640000000
69	5815	946.07720000000
70	6035	1019.06760000000
71	6256	1095.12040000000
72	6478	1174.27280000000
73	6701	1256.56160000000
74	6925	1342.02360000000
75	7150	1430.69600000000
76	7376	1522.61600000000
77	7603	1617.82120000000
78	7831	1716.34880000000
79	8060	1818.23600000000
80	8390	1970.21960000000

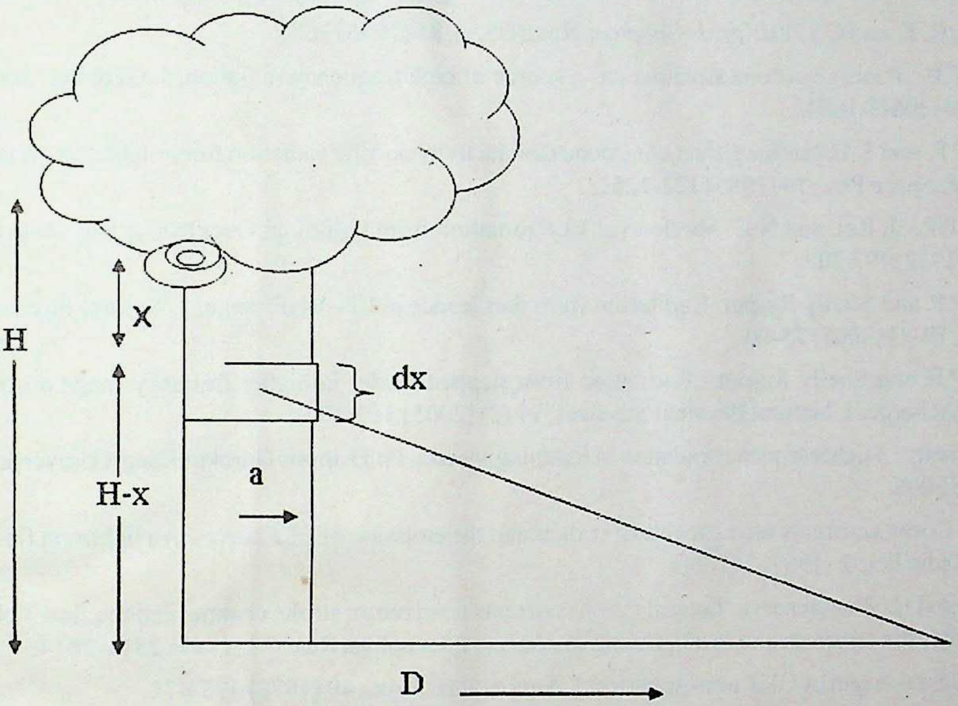

Fig. 1: A drawing defining the factor used in computing the electric field of a stepped leader.

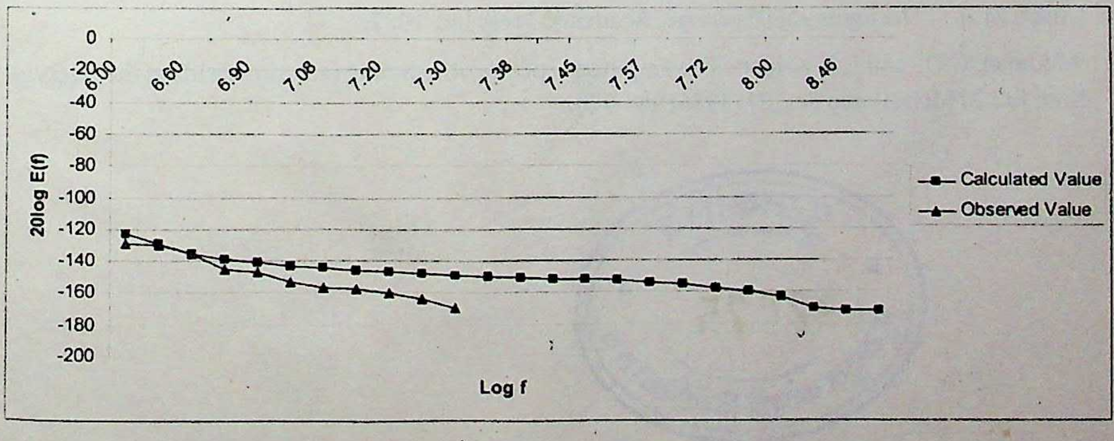

Fig. 2: Calculated and observed values of radiated fields from stepped leader

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